Year 12 Maths Transition Pack

This pack is designed to help you review your GCSE Mathematics to ensure you are ready for the A-Level course in September.

This pack contains a selection of GCSE topics that are an essential foundation to the A-Level course. You are expected to use some of these questions and helpful links, as needed, to help you revise these topics.

Once you are happy with the content, please complete the Diagnostic test on page 45. You should mark this test, make corrections and complete some additional work where needed using the support videos suggested in this booklet. You MUST bring your marked Diagnostic test with you to your first A-Level Maths lesson.

Contents

Indices

Basic rules of indices

1 $a^m \times a^n = a^{m+n}$

$5^2 \times 5^3 = (5 \times 5) \times (5 \times 5 \times 5) = 5^5$

or $5^2 \times 5^3 = 5^{2+3} = 5^5$

2 $a^m \div a^n = a^{m-n}$

$4^5 \div 4^2 = \dfrac{4.4.4.\not{4}.\not{4}}{\not{4}.\not{4}} = 4^3$

or $4^5 \div 4^2 = 5^{5-2} = 4^3$

3 $(a^m)^n = a^{mn}$

$(6^2)^3 = (6 \times 6) \times (6 \times 6) \times (6 \times 6) = 6^6$

or $(6^2)^3 = 6^{2 \times 3} = 6^6$

4 $a^{-n} = \dfrac{1}{a^n}$ consider $7^2 \div 7^4 = \dfrac{\not{7}\not{7}}{7.7.\not{7}.\not{7}} = \dfrac{1}{7^2} = 7^{-2}$

In particular, $a^{-1} = \dfrac{1}{a}$. The *reciprocal* of a is $\dfrac{1}{a}$.

5 $a^{\frac{1}{n}}$ means 'the nth root of a'

$9^{\frac{1}{2}} = \sqrt[2]{9} = 3$

$16^{\frac{1}{4}} = \sqrt[4]{16} = 2$

Consider $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = \sqrt{5} \times \sqrt{5} = 5$

Using rule 1: $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1$

6 $a^{\frac{m}{n}}$ means 'the nth root of a raised to the power m'.

$16^{\frac{3}{2}} = (\sqrt{16})^3 = 4^3 = 64$

7 $a^0 = 1$ for any value of a which is not zero.

Consider $7^3 \div 7^3 = 7^{3-3} = 7^0 = 1$.

Example 1

Simplify the following:

a $y^3 \times y^2 = y^{3+2} = y^5$

b $a^7 \div a^4 = a^{7-4} = a^3$

c $(p^3)^4 = p^{12}$

d $(2a^2)^3 = 2^3 \times (a^2)^3 = 8a^6$

e $2m^3 \times 5m^2 = 10m^5$

Example 2

Evaluate the following:

a $1000^{\frac{1}{3}} = \sqrt[3]{1000} = 10$

b $7^{-1} = \dfrac{1}{7}$

c $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

d $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

e $25^{-\frac{3}{2}} = \dfrac{1}{25^{\frac{3}{2}}} = \dfrac{1}{(\sqrt{25})^3} = \dfrac{1}{5^3} = \dfrac{1}{125}$

Extra Support

Laws of Indices - https://corbettmaths.com/2013/03/13/laws-of-indices-algebra/
Fractional Indices - https://corbettmaths.com/2013/03/03/fractional-indices/
Negative Indices - https://corbettmaths.com/2013/03/24/negative-indices/

Exercise 1A

Express in index form:

1 $2 \times 2 \times 2 \times 2$	**2** $7 \times 7 \times 7 \times 7 \times 7 \times 7$	**3** $2 \times 2 \times 6 \times 6 \times 6$
4 $\frac{1}{8 \times 8 \times 8}$	**5** $\frac{1}{7}$	**6** $\sqrt{11}$
7 $\sqrt[3]{5}$	**8** $(\sqrt{7})^5$	**9** $(n \times n \times n)^3$

Evaluate the following:

10 $2^2 \times 2^2$	**11** 50^0	**12** 4^{-2}
13 $27^{\frac{1}{3}}$	**14** $1^{\frac{2}{3}}$	**15** $2^3 \div 2^{-1}$
16 $(\sqrt{9})^3$	**17** $(1000\,000)^{\frac{1}{6}}$	**18** $5^{-1} \times 6^0$

Simplify the following:

19 $x^3 \times x^4$	**20** $y^6 \times y^5$	**21** $z^7 \times z^2$
22 $\frac{a^{12}}{a^5}$	**23** $(r^7)^2$	**24** $(d^3)^4$
25 $(s^3)^3 \times (s^2)^9$	**26** $(t^7)^3 \div (t^5)^2$	**27** $(3a^2b^3)^3$
28 $(k^{\frac{1}{2}})^6$	**29** $w^{-2} \div w$	**30** $(x^{-5})^{-2}$

Evaluate the following:

31 $2^{-1} + 4^{-1}$	**32** $(0.04)^{\frac{1}{2}}$	**33** $1000^{\frac{4}{3}}$
34 $(\frac{1}{9})^{-\frac{1}{2}}$	**35** $3^{-2} \times 36^{\frac{1}{2}}$	**36** $8^{\frac{1}{3}} + 8^0$

Simplify the following:

38 $(a^3)^3$	**39** $c^7 \div c$	**40** $e^4 \div e^{-2}$
41 $2x^3 \times 5xy$	**42** $10t^4u^2 \div 2t^2$	**43** $(v^4)^{-\frac{1}{2}}$
44 $12x^4y^2z \div 4x^3y^2$	**45** $(36m^4n^2)^{\frac{1}{2}}$	**46** $3p^5q^3 \times 5q^{-2}r^4$
47 $(8a^6b^3)^{\frac{1}{3}}$	**48** $(16x^8y^{12})^{\frac{1}{4}} \times (3xy^3)$	**49** $\frac{(2r^2)^5 \times (3r^4)^3}{(6r^3)^2}$

Answers

1 2^4	**2** 7^6	**3** $2^2 \times 6^3$
4 8^{-3}	**5** 7^{-1}	**6** $11^{\frac{1}{2}}$
7 $5^{\frac{1}{3}}$	**8** $7^{\frac{5}{2}}$	**9** n^9
10 16	**11** 1	**12** $\frac{1}{16}$
13 3	**14** 1	**15** 16
16 27	**17** 10	**18** $\frac{1}{5}$
19 x^7	**20** y^{11}	**21** z^9
22 a^7	**23** r^{14}	**24** d^{12}
25 s^{27}	**26** t^{11}	**27** $27a^6b^9$
28 k^3	**29** w^{-3}	**30** x^{10}
31 $\frac{3}{4}$	**32** 0.2	**33** 10 000
34 3	**35** $\frac{2}{3}$	**36** 3

38 a^9	**39** c^6	**40** e^6
41 $10x^4y$	**42** $5t^2u^2$	**43** $\frac{1}{v^2}$
44 $3xz$	**45** $6m^2n$	**46** $15p^5qr^4$
47 $2a^2b$	**48** $48x^3y^6$	**49** $24r^{16}$
50 a 1.99	**b** 0.94	**c** 2.24
d 2.78	**e** 2.65	**f** 6.75

Equations with Indices

Example 3

Solve the equation $2^n = 16$.

We use the principle that if

$a^x = a^y$ then $x = y$

In this case $16 = 2^4$

so $2^n = 2^4$

$n = 4$

Example 4

Solve the equation $3^{2x-1} = 27$

We have $3^{2x-1} = 3^3$

so $2x - 1 = 3$

$x = 2$

Example 5

Solve the equation $8^{n+1} = 16^n$.

We must write $8 = 2^3$ and $16 = 2^4$

So $(2^3)^{n+1} = (2^4)^n$

$2^{3n+3} = 2^{4n}$

$3n + 3 = 4n$

$n = 3$

Example 6

Solve the equation $x^{\frac{2}{3}} = \frac{1}{4}$.

Cube both sides of the equation,

$x^2 = (\frac{1}{4})^3 = \frac{1}{64}$

Take the square root, $x = \frac{1}{8}$

EXERCISE 1B

Solve the equations for x.

1 $2^x = 8$ **2** $5^x = 25$ **3** $4^x = \frac{1}{4}$

4 $3^x = \frac{1}{27}$ **5** $3^x = 81$ **6** $8^x = 1$

7 $10^x = 0.1$ **8** $10^x = 100\,000$ **9** $2^{x-2} = 16$

10 Write the following as powers of 2.

a $2\sqrt{2}$ **b** $\sqrt{16}$ **c** $\sqrt[3]{16}$

d 64 **e** $\dfrac{1}{32}$ **f** $(\sqrt{2})^3$

g $\sqrt[5]{2}$ **h** 1024 **i** $\sqrt{32}$

j 1 **k** $\left(\dfrac{1}{512}\right)^2$ **l** 4^8

11 Solve the following equations:

a $32^n = 16$ **b** $128^n = 8$ **c** $27^n = 9$

d $\left(\frac{1}{4}\right)^n = 8$ **e** $\left(\frac{125}{8}\right)^n = \frac{25}{4}$ **f** $\left(\frac{81}{16}\right)^n = \frac{32}{243}$

12 Find the value of n (as a fraction or whole number) in the following equations:

a $27^{n-1} = 3^6$ **b** $8^{2n+1} = 32^{n+1}$

c $81^{2n-1} = 27^{3n+1}$ **d** $16^{2-n} = \left(\frac{1}{4}\right)^{n+1}$

13 Solve the following equations, giving your answers to 2 s.f., where necessary:

a $x^2 = 30$ **b** $x^5 = 123$ **c** $x^{\frac{1}{2}} = \frac{1}{11}$

d $x^{\frac{1}{4}} = 3$ **e** $x^{-1} = 6$ **f** $x^{\frac{2}{3}} = 4$

g $x^{-3} = 7$ **h** $x^{-\frac{1}{3}} = 13$ **i** $x^{-\frac{3}{5}} = \frac{1}{3}$

14 Solve the following equations, showing your working clearly (by first rearranging to give an equation of the form $x^n = c$ and then solving for x):

a $x^3 = 64$ **b** $3x^{-1} = 6$ **c** $3x^{\frac{1}{2}} = 15$

d $2x^3 - 54 = 0$ **e** $5x^{-2} - 80 = 0$ **f** $2x^{\frac{2}{3}} - 1 = 17$

g $4x^{\frac{3}{2}} = 36x^{\frac{1}{2}}$ **h** $81x^{\frac{1}{3}} = x^{-1}$ **i** $3x^{\frac{3}{5}} = 48x^{-\frac{1}{5}}$

Answers

1 3 **2** 2 **3** -1

4 -3 **5** 4 **6** 0

7 -1 **8** 5 **9** 6

10 a $2^{\frac{3}{2}}$ **b** 2^2 **c** $2^{\frac{4}{3}}$

d 2^6 **e** 2^{-5} **f** $2^{\frac{3}{2}}$

g $2^{\frac{1}{5}}$ **h** 2^{10} **i** $2^{\frac{5}{2}}$

j 2^0 **k** 2^{-18} **l** 2^{16}

11 a $\frac{4}{5}$ **b** $\frac{3}{7}$ **c** $\frac{2}{3}$

d $-\frac{3}{2}$ **e** $\frac{2}{3}$ **f** $-\frac{5}{4}$

More Equations

Example 7

Solve the equation $20^x \times 5^{3x} = 50$.

Write 20 and 50 in terms of their prime factors.

So $20 = 2^2 \times 5$

and $50 = 2 \times 5^2$

We have
$$(2^2 \times 5)^x \times 5^{3x} = 2 \times 5^2$$
$$2^{2x} \times 5^x \times 5^{3x} = 2 \times 5^2$$
$$2^{2x} \times 5^{4x} = 2^1 \times 5^2$$

So
$$2x = 1$$
$$x = \tfrac{1}{2} \qquad [\text{check } 5^{4x} = 5^2, \text{ with } x = \tfrac{1}{2}\checkmark]$$

Example 8

Express in the form x^n: **a** $\left(\frac{\sqrt{x}}{\sqrt[3]{x}}\right)^2$ **b** $\frac{x}{\sqrt[5]{x^2}}$

a $\left(\frac{\sqrt{x}}{\sqrt[3]{x}}\right)^2 = \left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}\right)^2$

$= \frac{x}{x^{\frac{2}{3}}}$

$= x^{\frac{1}{3}}$

b $\frac{x}{\sqrt[5]{x^2}} = \frac{x^1}{x^{\frac{2}{5}}}$

$= x^{\frac{3}{5}}$

Practice

5 Express the following in the form x^n where n is to be determined:

a $x^2\sqrt{x}$ **b** $\sqrt[5]{x^3}$ **c** $\frac{\sqrt{x}}{x}$

d $x(\sqrt[3]{x^2})$ **e** $\frac{\sqrt[3]{x}}{\sqrt[4]{x}}$ **f** $(\sqrt[3]{x})^2$

6 Solve the following:

a $x\sqrt{x} = 8$ **b** $\frac{x^2}{\sqrt{x}} = 27$ **c** $5\sqrt{x} = x$

d $\sqrt[3]{x^2} = 25$ **e** $(\sqrt[3]{x})^2 = 9$ **f** $\frac{x\sqrt{x}}{\sqrt[3]{x}} = 128$

Answers

5 a $5 + 4\sqrt{2}$ **b** $3 + \sqrt{5}$ **c** $3 + 2\sqrt{2}$
d $15 - 6\sqrt{7}$ **e** $8 + 2\sqrt{15}$ **f** $7 - 4\sqrt{3}$
g 1 **h** $-1 + \sqrt{22}$ **i** 8
6 a $30\sqrt{5}$ **b** $6\sqrt{2}$ **c** $30\sqrt{10}$
d $12\sqrt{7}$ **e** $33\sqrt{6}$ **f** $56\sqrt{5}$

Surds

Numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{11}$ are called surds.

A surd is an *irrational* number which means that it cannot be written exactly as a decimal number or in the form $\frac{a}{b}$ where a and b are integers.

When an answer is given using a surd it is an *exact* answer.

The following rules apply:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Multiplying surds: Multiply out brackets in the usual way and collect similar terms.

For example:

a $(5 - \sqrt{3})(7 + \sqrt{3}) = 35 - 3 + 5\sqrt{3} - 7\sqrt{3}$
$= 32 - 2\sqrt{3}$

b $(\sqrt{2} + 1)(\sqrt{2} + 5) = 2 + \sqrt{2} + 5\sqrt{2} + 5$
$= 7 + 6\sqrt{2}$

Dividing surds: Multiply top and bottom by an appropriate number so that no surds remain in the denominator.

A *common mistake* occurs with surds.

$\sqrt{4}$ means 'the positive square root of 4'.

So $\sqrt{4} = 2$ only and *not* ± 2.

Notice that the solutions of the equation $x^2 = 4$ are $x = 2, -2$.

You *can* write $x = +\sqrt{4}$ or $-\sqrt{4}$.

Example 1

Simplify the following:

a $\sqrt{75} = \sqrt{(25 \times 3)}$
$= \sqrt{25} \times \sqrt{3}$
$= 5\sqrt{3}$

b $\sqrt{12} = \sqrt{(4 \times 3)}$
$= \sqrt{4} \times \sqrt{3}$
$= 2\sqrt{3}$

c $\frac{\sqrt{27}}{\sqrt{3}} = \sqrt{\frac{27}{3}}$
$= \sqrt{9}$
$= 3$

d $\sqrt{5} + 2\sqrt{5} = 3\sqrt{5}$

e $\sqrt{3} + \sqrt{12} = \sqrt{3} + \sqrt{(4 \times 3)}$
$= \sqrt{3} + 2\sqrt{3}$
$= 3\sqrt{3}$

Example 2

Rationalise the denominator of the fraction $\frac{10}{\sqrt{5}}$.

The fraction $\frac{10}{\sqrt{5}}$ can be written with a rational denominator by multiplying numerator and denominator by $\sqrt{5}$.

$$\frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{\sqrt{5} \times \sqrt{5}}$$

[Notice that this does not change the fraction.]

$$= \frac{10\sqrt{5}}{5}$$
$$= 2\sqrt{5}$$

Example 3

Rationalise the denominator of $\frac{2}{3+\sqrt{2}}$.

Multiply both the top and the bottom of the fraction by $3-\sqrt{2}$.
[Notice the change of sign from + to −.]

$$\frac{2}{3+\sqrt{2}} = \frac{2(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$$

$$= \frac{2(3-\sqrt{2})}{9-2} = \frac{2(3-\sqrt{2})}{7}$$

Example 4

Rationalise the denominator of $\frac{14+2\sqrt{5}}{3-\sqrt{5}}$.

$\frac{14+2\sqrt{5}}{3-\sqrt{5}}$ Multiply top and bottom by $(3+\sqrt{5})$.

$$\frac{(14+2\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{42+6\sqrt{5}+14\sqrt{5}+10}{9-5} = \frac{52+20\sqrt{5}}{4} = 13+5\sqrt{5}$$

Extra Support

Simplifying Surds - https://corbettmaths.com/2013/05/11/surds/

Adding and Subtracting - https://corbettmaths.com/2013/05/11/surds-addition/

Rationalising - https://corbettmaths.com/2013/05/11/rationalising-denominators/

Expanding Brackets - https://corbettmaths.com/2013/05/11/rationalising-denominators/

Answers for Exercise 1D (Qs on next page)

1 a $3\sqrt{3}$	**b** $5\sqrt{5}$	**c** $2\sqrt{7}$
d $6\sqrt{3}$	**e** $7\sqrt{3}$	**f** $11\sqrt{2}$
2 a/f, b/h, c/e, d/g		
3 a $3\sqrt{2}$	**b** $3\sqrt{7}$	**c** $2\sqrt{2}$
d $9\sqrt{3}$	**e** $5\sqrt{7}$	**f** $10\sqrt{5}$
4 a T	**b** T	**c** F
d T	**e** F	**f** T
g F	**h** T	**i** T
5 a $5+4\sqrt{2}$	**b** $3+\sqrt{5}$	**c** $3+2\sqrt{2}$
d $15-6\sqrt{7}$	**e** $8+2\sqrt{15}$	**f** $7-4\sqrt{3}$
g 1	**h** $-1+\sqrt{22}$	**i** 8
6 a $30\sqrt{5}$	**b** $6\sqrt{2}$	**c** $30\sqrt{10}$
d $12\sqrt{7}$	**e** $33\sqrt{6}$	**f** $56\sqrt{5}$
7 a $4\sqrt{2}$	**b** $\sqrt{5}$	**c** $5\sqrt{3}$
d $4\sqrt{2}$	**e** $9\sqrt{3}$	**f** $10\sqrt{5}$
g $6\sqrt{5}$	**h** $\sqrt{6}$	
8 a $3\sqrt{2}-3$	**b** $2\sqrt{5}+2$	**c** $3\sqrt{7}+6$
d $\sqrt{5}-\sqrt{3}$	**e** $2\sqrt{7}+2\sqrt{2}$	**f** $\sqrt{13}+3$

EXERCISE 1D

1 Simplify the following as far as possible:

a $\sqrt{27}$ **b** $\sqrt{125}$ **c** $\sqrt{28}$

d $\sqrt{108}$ **e** $\sqrt{147}$ **f** $\sqrt{242}$

2 Sort these into four pairs of equal value.

a $\sqrt{30}$ **b** $\sqrt{12}$ **c** $\sqrt{10}$ **d** 5

e $\dfrac{\sqrt{20}}{\sqrt{2}}$ **f** $\sqrt{10} \times \sqrt{3}$ **g** $\dfrac{\sqrt{75}}{\sqrt{3}}$ **h** $2\sqrt{3}$

3 Write the following in the form $a\sqrt{b}$ in its simplest form:

a $\sqrt{2} + 2\sqrt{2}$ **b** $5\sqrt{7} - 2\sqrt{7}$ **c** $\sqrt{32} - 2\sqrt{2}$

d $\sqrt{75} + 4\sqrt{3}$ **e** $3\sqrt{7} + \sqrt{28}$ **f** $3\sqrt{5} + \sqrt{245}$

4 Answer 'True' or 'False'.

a $\sqrt{20} = 2\sqrt{5}$ **b** $(\sqrt{7})^2 = 7$ **c** $\sqrt{9} = \pm 3$

d $\sqrt{\dfrac{18}{6}} = \sqrt{3}$ **e** $\sqrt{3} + \sqrt{3} = \sqrt{6}$ **f** $(\sqrt{2})^4 = 4$

g $\sqrt{4+9} = \sqrt{4} + \sqrt{9}$ **h** $\dfrac{\sqrt{8}}{2} = \sqrt{2}$ **i** $\sqrt{10-1} = 3$

5 Multiply out the following brackets and write in the form $a + b\sqrt{c}$ in its simplest form:

a $(\sqrt{2} + 1)(\sqrt{2} + 3)$ **b** $(\sqrt{5} + 2)(\sqrt{5} - 1)$ **c** $(1 + \sqrt{2})^2$

d $(\sqrt{7} - 2)(\sqrt{7} - 4)$ **e** $(\sqrt{3} + \sqrt{5})^2$ **f** $(2 - \sqrt{3})^2$

g $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$ **h** $(\sqrt{11} - 2\sqrt{2})(\sqrt{11} + 3\sqrt{2})$ **i** $(\sqrt{18} - \sqrt{2})^2$

6 Write the following in the form $a\sqrt{b}$ in its simplest form:

a $3\sqrt{2} \times 5\sqrt{10}$ **b** $\sqrt{3} \times 2\sqrt{6}$ **c** $2\sqrt{6} \times 5\sqrt{15}$

d $3\sqrt{8} \times \sqrt{14}$ **e** $3\sqrt{22} \times \sqrt{33}$ **f** $2\sqrt{8} \times 7\sqrt{10}$

7 Write the following in the form $a\sqrt{b}$ in its simplest form:

a $\dfrac{8}{\sqrt{2}}$ **b** $\dfrac{10}{\sqrt{5}}$ **c** $\dfrac{15}{\sqrt{3}}$ **d** $\dfrac{16}{\sqrt{8}}$

e $\dfrac{81}{\sqrt{27}}$ **f** $\dfrac{50\sqrt{2}}{\sqrt{10}}$ **g** $\dfrac{6\sqrt{15}}{\sqrt{3}}$ **h** $\dfrac{3\sqrt{14}}{\sqrt{21}}$

8 Write the following in the form $a + b\sqrt{c}$ in its simplest form:

a $\dfrac{3}{\sqrt{2} + 1}$ **b** $\dfrac{8}{\sqrt{5} - 1}$ **c** $\dfrac{9}{\sqrt{7} - 2}$

Answers – on previous page

Polynomials

A polynomial is an algebraic expression which is the sum of a number of terms. Here is an example:

$$x^3 + 2x^2 - 8x + 11$$

This polynomial has four terms.

The *degree* of the above polynomial is 3. This means that the highest power of x is 3.

The degree of the polynomial $3x^6 - 7x^2 + x$ is 6.

In $3x^6 - 7x^2 + x$ the *coefficient* of x^6 is 3, the coefficient of x^2 is -7 and the coefficient of x is 1.

Adding and subtracting

When adding (or subtracting) two polynomials we simply add together (or subtract) the coefficients of corresponding terms.

For example: **a** $(x^2 - 2x + 7) + (3x^2 + 5x + 6) = 4x^2 + 3x + 13$

b $(x^3 + 4x + 4) - (x^2 + x + 1) = x^3 - x^2 + 3x + 3$

Multiplying

When multiplying two polynomials we multiply every term in one polynomial by every term in the other polynomial.

For example: **a** $(x + 7)(2x + 1) = x(2x + 1) + 7(2x + 1)$

$$= 2x^2 + x + 14x + 7$$
$$= 2x^2 + 15x + 7$$

b $(x + 4)(x^3 + 2x^2 + 5x + 3) = x(x^3 + 2x^2 + 5x + 3) + 4(x^3 + 2x^2 + 5x + 3)$

$$= x^4 + 2x^3 + 5x^2 + 3x + 4x^3 + 8x^2 + 20x + 12$$
$$= x^4 + 6x^3 + 13x^2 + 23x + 12$$

Factorising

The process of writing an expression as a product of its factors is called factorisation.

Examples: **a** $a^2 - b^2 = (a - b)(a + b)$

b $x^2 + 4x + 3 = (x + 1)(x + 3)$

In this section we will factorise quadratic expressions only. The factorisation of cubic expressions, such as $x^3 - 4x^2 + 3$, is dealt with in Part 6.

When factorising a quadratic expression, such as $ax^2 + bx + c$, we are looking to write it in the form $(mx + n)(px + q)$. We need to consider two cases, when $a = 1$ and when $a \neq 1$.

Example 1

Factorise $x^2 + 8x + 15$.

Step 1: Find two numbers whose product is 15 and whose sum is 8, i.e. 3 and 5.

Step 2: Express $x^2 + 8x + 15$ as $(x + 3)(x + 5)$.

Example 2

Factorise $x^2 - x - 12$.

Step 1: Find two numbers whose product is -12 and whose sum is -1, i.e. -4 and 3.

Step 2: Express $x^2 - x - 12$ as $(x - 4)(x + 3)$.

Example 3

Factorise $2x^2 + 7x + 6$.

Step 1: Find two numbers whose product is 12 (i.e. 2×6) and whose sum is 7. Here the numbers are 4 and 3.

Step 2: Rewrite $2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6$.

Step 3: Factorise this as two separate linear expressions
$2x^2 + 4x + 3x + 6 = 2x(x + 2) + 3(x + 2)$.

Step 4: Note that $(x + 2)$ is a common factor and use this to give
$2x^2 + 7x + 6 = (2x + 3)(x + 2)$.

Example 4

Factorise $3x^2 - 14x + 8$.

Step 1: Find two numbers whose product is 24 (i.e. 3×8) and whose sum is -14. The numbers are -12 and -2.

Step 2: Rewrite $3x^2 - 14x + 8 = 3x^2 - 12x - 2x + 8$.

Step 3: Factorise this as two separate linear expressions, so
$3x^2 - 12x - 2x + 8 = 3x(x - 4) - 2(x - 4)$.

Step 4: Note that there is a common linear factor and use this to give
$3x^2 - 14x + 8 = (3x - 2)(x - 4)$.

Example 5

Factorise **a** $2x^2 - 3x$ **b** $x^3 - 5x^2 + 6x$ **c** $4x^2 - 1$

a $2x^2 - 3x = x(2x - 3)$

b $x^3 - 5x^2 + 6x = x(x^2 - 5x + 6)$

$= x(x - 3)(x - 2)$

Extra Support
Expanding two brackets - https://corbettmaths.com/2013/12/23/expanding-two-brackets-video-14/

Expanding three brackets - https://corbettmaths.com/2013/12/27/expanding-three-brackets-video-15/

Factorising Quadratics - https://corbettmaths.com/2013/02/06/factorising-quadratics-1/
https://corbettmaths.com/2019/03/26/splitting-the-middle-term/

Factorising – Difference of two squares - Difference between two squares Video – Corbettmaths

Exercise 1E

1 Simplify the following:

a $(x^2 + 7x + 1) + (4x^2 - x + 5)$
b $(5x^2 - 3x + 4) + (x^2 + 6x - 10)$
c $(x^3 + x^2 + 4x + 1) + (2x^3 - x^2 + 6x + 3)$
d $(5x^2 + 7x + 1) - (x^2 + 2x + 3)$
e $(6x^3 + 4x + 5) - (x^3 + 2x^2 + 3x + 1)$
f $(9x^4 + 3x^2 + 4) - (3x^4 + x + 1)$
g $(x^2 - 3x + 2) + (3x^2 + 7x - 10) + (x^2 - 3x + 1)$

2 Expand the following linear brackets:

a $(x + 3)(x + 1)$ **b** $(z + 5)(z - 2)$ **c** $(r - 1)(r - 3)$
d $(y - 3)(y + 3)$ **e** $(5x - 1)(2x - 3)$ **f** $(7x + 2)^2$
g $(3x - 1)(2x - 1)$ **h** $(5y - 1)(7y + 3)$ **i** $(7u + 1)(2u + 1)$
j $(3k - 4)(3k + 4)$

3 Expand the following brackets:

a $(x + 3)(x^2 + 3x + 1)$ **b** $(r - 3)(r^2 - 2r + 5)$
c $(2t - 3)(t^2 + 3t + 1)$ **d** $(2w - 1)(3w^2 - w + 1)$
e $(5y + 1)(2y^2 + 5y - 1)$ **f** $(5z^2 + 4z + 1)(2z^2 - 3z - 2)$

4 Find the following terms in the following expansions:

a the x^2 term in $(x+1)(x^2-2x+3)$

b the y term in $(2y-1)(y^2+3y+4)$

c the z^3 term in $(z-2)(5z^3+2z^2-3z-5)$

d the p^2 term in $(p^2+3p+1)(2p^2-5p+4)$

e the r^3 term in $(2r^3+3r^2+r-4)(r^3-2r^2+3r-1)$

5 If $f(x)=x^3+2x^2+3x+1$ and $g(x)=2x^3-x^2+2x-3$ then find the following:

a the x^2 term in $2f(x)+3g(x)$ **b** the x term in $5f(x)-2g(x)$

c the x^3 term in $f(x)g(x)$

6 Find a, b and c in the following:

a $x^3-2x^2+5x-4=(x-1)(ax^2+bx+c)$

b $x^3+x^2-5x-2=(x-2)(ax^2+bx+c)$

c $2x^3+5x^2+9=(x+3)(ax^2+bx+c)$

d $3x^3-15x^2-2x+10=(x-5)(ax^2+bx+c)$

e $2x^3+11x^2+9x+2=(2x+1)(ax^2+bx+c)$

7 Factorise the following:

a $x^2+7x+12$ **b** $x^2+10x+21$ **c** $x^2+2x-15$ Scre

d x^2-4x-5 **e** $x^2+10x+25$ **f** x^2+x-6

Answers

1 a $5x^2+6x+6$ **b** $6x^2+3x-6$
c $3x^3+10x+4$ **d** $4x^2+5x-2$
e $5x^3-2x^2+x+4$ **f** $6x^4+3x^2-x+3$
g $5x^2+x-7$

2 a x^2+4x+3 **b** $z^2+3z-10$
c r^2-4r+3 **d** y^2-9
e $10x^2-17x+3$ **f** $49x^2+28x+4$
g $6x^2-5x+1$ **h** $35y^2+8y-3$
i $14u^2+9u+1$ **j** $9k^2-16$

3 a $x^3+6x^2+10x+3$ **b** $r^3-5r^2+11r-15$
c $2t^3+3t^2-7t-3$ **d** $6w^3-5w^2+3w-1$
e $10y^3+27y^2-1$
f $10z^4-7z^3-20z^2-11z-2$

4 a $-x^2$ **b** $5y$ **c** $-8z^3$
d $-9p^2$ **e** r^3

5 a x^2 **b** $11x$ **c** $0x^3$

6 a $(1x^2-x+4)$ **b** (x^2+3x+1)
c $(2x^2-x+3)$ **d** $(3x^2+0x-2)$
e (x^2+5x+2)

7 a $(x+3)(x+4)$ **b** $(x+3)(x+7)$
c $(x+5)(x-3)$ **d** $(x-5)(x+1)$
e $(x+5)^2$ **f** $(x+3)(x-2)$
g $(x-7)(x+4)$ **h** $(x-20)(x+12)$

8 a $(2x+3)(x+2)$ **b** $(3x+2)(x+4)$
c $(3a+4)(2a+1)$ **d** $(2y+9)(2y+1)$
e $(3d+4)(6d+1)$ **f** $(8z+2)(z+3)$
g $(3r-4)(4r+1)$ **h** $(3u-4)(5u+1)$
i $(5e-7)(5e+7)$ **j** $(4s-5)(4s+5)$

9 a $3x(x-5)$ **b** $(x-4)(x+4)$
c $x(x+5)(x+2)$ **d** $x(x^2+8x+10)$
e $x(x-1)(x+1)$ **f** $x(x-4)(x+4)$
g $a(a-b)(a+b)$ **h** $2x(2x-y)(2x+y)$
i $x(2x+3)(x+1)$

Solving Quadratics

Solving Quadratics by factorising

2.2 You can solve quadratic equations using factorisation.

Quadratic equations have two solutions or roots. (In some cases the two roots are equal.)
To solve a quadratic equation, put it in the form $ax^2 + bx + c = 0$.

Example 2

Solve the equation $x^2 = 9x$

$x^2 = 9x$

$x^2 - 9x = 0$ — Rearrange in the form $ax^2 + bx + c = 0$.

$x(x - 9) = 0$ — Factorise by x (factorising is in Chapter 1). Then either part of the product could be zero.

Then either $x = 0$

or $x - 9 = 0 \Rightarrow x = 9$

So $x = 0$ or $x = 9$ are the two solutions of the equation $x^2 = 9x$.

A quadratic equation has two solutions (roots). In some cases the two roots are equal.

Example 3

Solve the equation $x^2 - 2x - 15 = 0$

$x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$ — Factorise.

Then either $x + 3 = 0 \Rightarrow x = -3$

or $x - 5 = 0 \Rightarrow x = 5$

The solutions are $x = -3$ or $x = 5$.

Example 4

Solve the equation $6x^2 + 13x - 5 = 0$

$6x^2 + 13x - 5 = 0$

$(3x - 1)(2x + 5) = 0$ — Factorise.

Then either $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

The solutions are $x = \frac{1}{3}$ or $x = -\frac{5}{2}$.

The solutions can be fractions or any other type of number.

Example 5

Solve the equation $x^2 - 5x + 18 = 2 + 3x$

$x^2 - 5x + 18 = 2 + 3x$

$x^2 - 8x + 16 = 0$ — Rearrange in the form $ax^2 + bx + c = 0$.

$(x - 4)(x - 4) = 0$ — Factorise.

Then either $x - 4 = 0 \Rightarrow x = 4$

or $x - 4 = 0 \Rightarrow x = 4$

$\Rightarrow x = 4$

Here $x = 4$ is the only solution, i.e. the two roots are equal.

Example 6

Solve the equation $(2x - 3)^2 = 25$

$(2x - 3)^2 = 25$

$2x - 3 = \pm 5$

$2x = 3 \pm 5$

Then either $2x = 3 + 5 \Rightarrow x = 4$

or $2x = 3 - 5 \Rightarrow x = -1$

The solutions are $x = 4$ or $x = -1$.

This is a special case.
Take the square root of both sides.
Remember $\sqrt{25} = +5$ or -5.
Add 3 to both sides.

Extra Support

Solving quadratics - https://corbettmaths.com/2013/05/03/solving-quadratics-by-factorising/

Exercise 2B

Solve the following equations:

1 $x^2 = 4x$	**2** $x^2 = 25x$
3 $3x^2 = 6x$	**4** $5x^2 = 30x$
5 $x^2 + 3x + 2 = 0$	**6** $x^2 + 5x + 4 = 0$
7 $x^2 + 7x + 10 = 0$	**8** $x^2 - x - 6 = 0$
9 $x^2 - 8x + 15 = 0$	**10** $x^2 - 9x + 20 = 0$
11 $x^2 - 5x - 6 = 0$	**12** $x^2 - 4x - 12 = 0$
13 $2x^2 + 7x + 3 = 0$	**14** $6x^2 - 7x - 3 = 0$
15 $6x^2 - 5x - 6 = 0$	**16** $4x^2 - 16x + 15 = 0$
17 $3x^2 + 5x = 2$	**18** $(2x - 3)^2 = 9$
19 $(x - 7)^2 = 36$	**20** $2x^2 = 8$
21 $3x^2 = 5$	**22** $(x - 3)^2 = 13$
23 $(3x - 1)^2 = 11$	**24** $5x^2 - 10x^2 = -7 + x + x^2$
25 $6x^2 - 7 = 11x$	**26** $4x^2 + 17x = 6x - 2x^2$

Exercise 2B

1 $x = 0$ or $x = 4$	**2** $x = 0$ or $x = 25$
3 $x = 0$ or $x = 2$	**4** $x = 0$ or $x = 6$
5 $x = -1$ or $x = -2$	**6** $x = -1$ or $x = -4$
7 $x = -5$ or $x = -2$	**8** $x = 3$ or $x = -2$
9 $x = 3$ or $x = 5$	**10** $x = 4$ or $x = 5$
11 $x = 6$ or $x = -1$	**12** $x = 6$ or $x = -2$
13 $x = -\frac{1}{2}$ or $x = -3$	**14** $x = -\frac{1}{3}$ or $x = \frac{3}{2}$
15 $x = -\frac{2}{3}$ or $x = \frac{3}{2}$	**16** $x = \frac{3}{2}$ or $x = \frac{5}{2}$
17 $x = \frac{1}{3}$ or $x = -2$	**18** $x = 3$ or $x = 0$
19 $x = 13$ or $x = 1$	**20** $x = 2$ or $x = -2$
21 $x = \pm\sqrt{\frac{5}{3}}$	**22** $x = 3 \pm \sqrt{13}$
23 $x = \frac{1 \pm \sqrt{11}}{3}$	**24** $x = 1$ or $x = -\frac{7}{6}$
25 $x = -\frac{1}{2}$ or $x = \frac{7}{3}$	**26** $x = 0$ or $x = -\frac{11}{6}$

Quadratic Formula

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.

Examples

Example 7 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$a = 1, b = 6, c = 4$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**1** Identify a, b and c and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$	**2** Substitute $a = 1$, $b = 6$, $c = 4$ into the formula.
$x = \frac{-6 \pm \sqrt{20}}{2}$	**3** Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2.
$x = \frac{-6 \pm 2\sqrt{5}}{2}$	**4** Simplify $\sqrt{20}$. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
$x = -3 \pm \sqrt{5}$	**5** Simplify by dividing numerator and denominator by 2.
So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$	**6** Write down both the solutions.

Extra Support

Quadratic formula Video - https://corbettmaths.com/2015/03/19/deriving-the-quadratic-formula/

- https://www.youtube.com/watch?v=aeA5L-4SrmQ&list=PLxHVbxhSvleS6TaN5EqyV0mu1W35t33KL&index=28

Exercise 2E

Solve the following quadratic equations by using the formula, giving the solutions in surd form. Simplify your answers.

1 $x^2 + 3x + 1 = 0$

2 $x^2 - 3x - 2 = 0$

3 $x^2 + 6x + 6 = 0$

4 $x^2 - 5x - 2 = 0$

5 $3x^2 + 10x - 2 = 0$

6 $4x^2 - 4x - 1 = 0$

7 $7x^2 + 9x + 1 = 0$

8 $5x^2 + 4x - 3 = 0$

9 $4x^2 - 7x = 2$

10 $11x^2 + 2x - 7 = 0$

Answers

Exercise 2E

1 $\frac{-3 \pm \sqrt{5}}{2}$, -0.38 or -2.62

2 $\frac{+3 \pm \sqrt{17}}{2}$, -0.56 or 3.56

3 $-3 \pm \sqrt{3}$, -1.27 or -4.73

4 $\frac{5 \pm \sqrt{33}}{2}$, 5.37 or -0.37

5 $\frac{-5 \pm \sqrt{31}}{3}$, -3.52 or 0.19

6 $\frac{1 \pm \sqrt{2}}{2}$, 1.21 or -0.21

7 $\frac{-9 \pm \sqrt{53}}{14}$, -0.12 or -1.16

8 $\frac{-2 \pm \sqrt{19}}{5}$, 0.47 or -1.27

9 2 or $-\frac{1}{4}$

10 $\frac{-1 \pm \sqrt{78}}{11}$, 0.71 or -0.89

Completing the Square

2.3 You can write quadratic expressions in another form by completing the square.

$x^2 + 2bx + b^2 = (x + b)^2$
$x^2 - 2bx + b^2 = (x - b)^2$

These are both perfect squares.

To complete the square of the function $x^2 + 2bx$ you need a further term b^2. So the completed square form is

$x^2 + 2bx = (x + b)^2 - b^2$

Similarly

$x^2 - 2bx = (x - b)^2 - b^2$

Example 8

Complete the square for the expression $x^2 + 8x$

$x^2 + 8x$
$= (x + 4)^2 - 4^2$
$= (x + 4)^2 - 16$

$2b = 8$, so $b = 4$

In general

■ **Completing the square:** $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

$= (x + 6)^2 - 36$

b $2x^2 - 10x$
$= 2(x^2 - 5x)$
$= 2[(x - \frac{5}{2})^2 - (\frac{5}{2})^2]$
$= 2(x - \frac{5}{2})^2 - \frac{25}{2}$

Here the coefficient of x^2 is 2.
So take out the coefficient of x^2.
Complete the square on $(x^2 - 5x)$.
Use $b = -5$.

Example 10

Solve the equation $x^2 + 8x + 10 = 0$ by completing the square.

$x^2 + 8x + 10 = 0$	Check coefficient of $x^2 = 1$.
$x^2 + 8x = -10$	Subtract 10 to get LHS in the form $ax^2 + b$.
$(x + 4)^2 - 4^2 = -10$	Complete the square for $(x^2 + 8x)$.
$(x + 4)^2 = -10 + 16$	Add 4^2 to both sides.
$(x + 4)^2 = 6$	
$(x + 4) = \pm\sqrt{6}$	Square root both sides.
$x = -4 \pm \sqrt{6}$	Subtract 4 from both sides.

Then the solutions (roots) of $x^2 + 8x + 10 = 0$ are either $x = -4 + \sqrt{6}$ or $x = -4 - \sqrt{6}$.

Leave your answer in surd form as this is a non-calculator question.

Example 11

Solve the equation $2x^2 - 8x + 7 = 0$.

$2x^2 - 8x + 7 = 0$	The coefficient of $x^2 = 2$.
$x^2 - 4x + \frac{7}{2} = 0$	So divide by 2.
$x^2 - 4x = -\frac{7}{2}$	Subtract $\frac{7}{2}$ from both sides.
$(x - 2)^2 - (2)^2 = -\frac{7}{2}$	Complete the square for $x^2 - 4x$.
$(x - 2)^2 = -\frac{7}{2} + 4$	Add $(2)^2$ to both sides.
$(x - 2)^2 = \frac{1}{2}$	Combine the RHS.
$x - 2 = \pm\sqrt{\frac{1}{2}}$	Square root both sides.
$x = 2 \pm \frac{1}{\sqrt{2}}$	Add 2 to both sides.

So the roots are either

$x = 2 + \frac{1}{\sqrt{2}}$

or $x = 2 - \frac{1}{\sqrt{2}}$

Extra Support

Completing the square - https://corbettmaths.com/2013/12/29/completing-the-square-video-10/

Exercise 2D

Solve these quadratic equations by completing the square (remember to leave your answer in surd form):

1 $x^2 + 6x + 1 = 0$

2 $x^2 + 12x + 3 = 0$

3 $x^2 - 10x = 5$

4 $x^2 + 4x - 2 = 0$

5 $x^2 - 3x - 5 = 0$

6 $2x^2 - 7 = 4x$

7 $4x^2 - x = 8$

8 $10 = 3x - x^2$

9 $15 - 6x - 2x^2 = 0$

10 $5x^2 + 8x - 2 = 0$

Answers

Exercise 2D

1 $x = -3 \pm 2\sqrt{2}$

2 $x = -6 \pm \sqrt{33}$

3 $x = 5 \pm \sqrt{30}$

4 $x = -2 \pm \sqrt{6}$

5 $x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$

6 $x = 1 \pm \frac{3}{2}\sqrt{2}$

7 $x = \frac{1}{8} \pm \frac{\sqrt{129}}{8}$

8 No real roots

9 $x = -\frac{3}{2} \pm \frac{\sqrt{39}}{2}$

10 $x = -\frac{4}{5} \pm \frac{\sqrt{26}}{5}$

Simultaneous Equations

Solve the equations:

$$2x - y = 1$$
$$4x + 2y = -30$$

$y = 2x - 1$ — Rearrange an equation to get either $x = \ldots$ or $y = \ldots$ (here $y = \ldots$).

$4x + 2(2x - 1) = -30$ — Substitute this into the other equation (here in place of y).

$4x + 4x - 2 = -30$ — Solve for x.

$8x = -28$

$x = -3\frac{1}{2}$

$y = 2(-3\frac{1}{2}) - 1 = -8$ — Substitute $x = -3\frac{1}{2}$ into $y = 2x - 1$ to find the value of y.

So solution is $x = -3\frac{1}{2}$, $y = -8$.

Example 3

Solve the equations:

a $x + 2y = 3$
$x^2 + 3xy = 10$

b $3x - 2y = 1$
$x^2 + y^2 = 25$

a $x = 3 - 2y$

$(3 - 2y)^2 + 3y(3 - 2y) = 10$

$9 - 12y + 4y^2 + 9y - 6y^2 = 10$

$-2y^2 - 3y - 1 = 0$

$2y^2 + 3y + 1 = 0$

$(2y + 1)(y + 1) = 0$

$y = -\frac{1}{2}$ or $y = -1$

So $x = 4$ or $x = 5$

Solutions are $x = 4$, $y = -\frac{1}{2}$

and $x = 5$, $y = -1$

Rearrange the linear equation to get $x = \ldots$ or $y = \ldots$ (here $x = \ldots$).

Substitute this into the quadratic equation (here in place of x).
$(3 - 2y)^2$ means $(3 - 2y)(3 - 2y)$ (see Chapter 1).

Solve for y using factorisation.

Find the corresponding x-values by substituting the y-values into $x = 3 - 2y$.

There are two solution pairs. The graph of the linear equation (straight line) would intersect the graph of the quadratic (curve) at two points.

Extra Support

Linear Simultaneous Equations - https://www.youtube.com/watch?v=I2gl20Kaxng&list=PL0sgfvQGLNJujLLbZQPPdCKXNkRM7X_9c&index=3

https://www.youtube.com/watch?v=0XvTVZiQNiw&list=PL0sgfvQGLNJujLLbZQPPdCKXNkRM7X_9c&index=5

Quadratic Simultaneous Equations - https://www.youtube.com/watch?v=Pd14gmRwaYs

Exercise 1H

1 Solve the following simultaneous equations.

a $2x + 3y = 30$
$y = 3x - 1$

b $a = 3b + 1$
$7a + 2b = 30$

c $c = 7d - 2$
$3c - 4d = 11$

d $10p + 11q = 21$
$p = 5q - 4$

e $2a - 9b = 73$
$b = 3 - 2a$

f $4a + 3b = 123$
$2a = 3 + 5b$

g $10w + 7x = 41$
$5w = 3x + 1$

h $7t + 3s = 33$
$2t = s + 2$

i $3v + 7w = 26$
$3w = 18 + v$

j $6x - 5y = 27$
$3y = 1 - 5x$

2 Solve the following simultaneous equations (by the method of substitution):

a $xy = 2$
$y = x + 1$

b $y(x + 1) = 10$
$y = 2x + 3$

c $x^2 + 2y = 12$
$y = 3x - 2$

d $xy + x + y = -1$
$y = 3x + 1$

e $xy - 2y - x = 2$
$x + y = 7$

f $3x^2 + 2y^2 = 30$
$y + 2x = 1$

3 Solve the following simultaneous equations (by the method of substitution):

a $x^2 + 3xy + y^2 = 11$
$x + y = 3$

b $3x^2 - y^2 + 2xy = 15$
$x + 2y = 8$

c $4x^2 + 3y^2 + 2xy = 5$
$3x + y = 2$

d $x^2 - 3xy + y^2 = 59$
$2x + y = 8$

Answers

1 a (3, 8) **b** (4, 1) **c** (5, 1)
d (1, 1) **e** (5, −7) **f** (24, 9)
g (2, 3) **h** (4, 3) **i** (−3, 5)
j (2, −3)

2 a (1, 2) **b** (−3.5, −4)
c (2, 4) or (−8, −26) **d** (−1, −2) or $(-\frac{2}{3}, -1)$
e (4, 3) **f** (2, −3) or $(-\frac{14}{11}, \frac{39}{11})$

3 a (1, 2) or (2, 1) **b** (2, 3) or $(-\frac{62}{7}, \frac{59}{7})$
c (1, −1) or $(\frac{7}{25}, \frac{29}{25})$ **d** (5, −2) or $(\frac{1}{11}, \frac{86}{11})$

4 (2, 3), (−3, −2)

5 $(3x + 2y)(2x + y)$, $x = \frac{1}{2}$, $y = \frac{1}{4}$

Inequalities

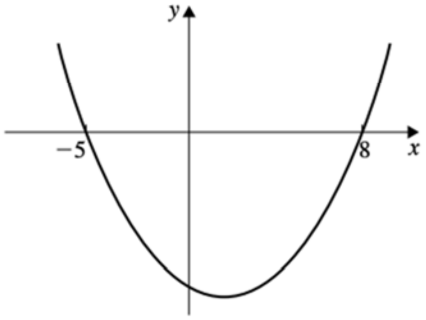
Example 1

Solve $x^2 - 3x - 40 < 0$.

We factorise to give $(x - 8)(x + 5) < 0$.

The critical values are 8 and -5.

Sketch the graph of $y = x^2 - 3x - 40$.



We are looking below the x-axis so we want *one* region, and hence *one* inequality.

So the solution is $-5 < x < 8$.

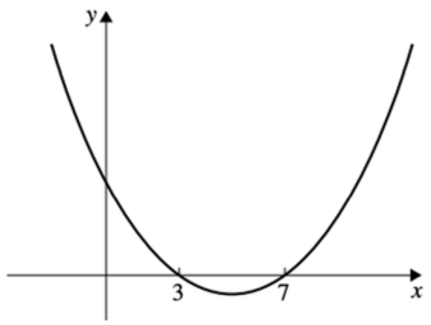
Example 2

Solve $x^2 - 10x + 21 \geqslant 0$.

Factorise to give $(x - 3)(x - 7) \geqslant 0$.

Critical values are 3 and 7.

Sketch the graph of $y = x^2 - 10x + 21$.



We are looking above x-axis so we want *two* regions, and hence *two* inequalities.

So the solution is $x \leqslant 3$ or $x \geqslant 7$.

Extra Support

Linear Inequalities - https://corbettmaths.com/2013/05/07/solving-inequalities-one-sign-corbettmaths/

Linear Inequalities - https://corbettmaths.com/2013/05/12/solving-inequalities-two-signs/

Quadratic Inequalities - https://corbettmaths.com/2016/08/07/quadratic-inequalities/

Exercise 1I

1 Solve the following inequalities:

a $2w + 1 \leqslant 5$

b $3 - 5x > 23$

c $3x + 1 > 2x + 6$

d $11x + 3 \leqslant 15x + 19$

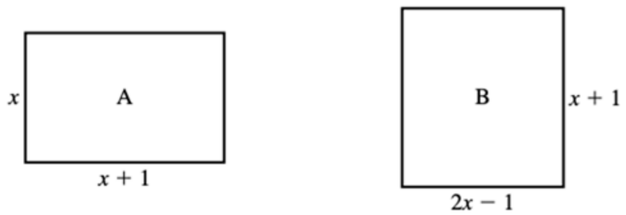
e $2 - x < 1$

f $4(3 - 2x) > 8$

g $9 < 5y - 1 < 24$

h $24 < 3 - 7z < 66$

2 The perimeter of rectangle B is greater than the perimeter of rectangle A.
Find the range of possible values of x.



3 Given that $3x > 1$ and $\frac{x}{4} \leqslant 1\frac{1}{4}$, list the possible integer values of x.

4 State the smallest integer n for which $5n > 31$.

5 What integer values of x satisfy $100 < 4^x < 10\,000$?

6 **a** The solution of $x^2 < 1$ is $-1 < x < 1$.

b Copy and complete

i If $x^2 < 9$, then $\square < x < \square$.

ii If $x^2 < 25$, then $\square < x < \square$.

iii If $x^2 > 49$, then $x > \square$ or $x < \square$.

7 Solve the following inequalities:

a $x^2 < 1$

b $x^2 > 16$

c $2x^2 - 1 \leqslant 17$

d $3x^2 - 2 < 46$

e $28 - x^2 < 3$

f $15 - 2x^2 \leqslant -17$

8 Solve the following inequalities (by first factorising the quadratic).

a $x^2 - 4x - 12 \geqslant 0$

b $x^2 + 6x + 8 < 0$

c $x^2 - 5x - 6 \geqslant 0$

d $x^2 + 5x + 6 < 0$

e $x^2 - 7x + 12 \geqslant 0$

f $x^2 - 3x + 2 \leqslant 0$

g $2x^2 - 11x + 12 < 0$

h $3x^2 - 13x - 8 < 2$

i $4x^2 + 16x - 6 < 3$

j $3x^2 - 13x + 18 \geqslant 6$

k $4x + 3 - 4x^2 \leqslant 0$

l $4 + 7x - 3x^2 > -22$

m $9x - 2x^2 > 10$

n $3 - 6x^2 < 8 - 17x$

9 The area of the rectangle must be greater than the area of the triangle.
Find the range of possible values of x.

10 Find the range of values of x which satisfy both of the inequalities *simultaneously*.

$$x^2 - 3x - 4 \geqslant 0 \quad \text{and} \quad 3x + 1 > 2x + 6$$

11 Find the range of values of x which satisfy both of the inequalities simultaneously.

$$x^2 - x - 6 \geqslant 0 \quad \text{and} \quad 5x + 1 < 2x - 8$$

Answers

1 a $w \leqslant 2$ **b** $x < -4$
c $x > 5$ **d** $x \geqslant -4$
e $x > 1$ **f** $x < \frac{1}{2}$
g $2 < y < 5$ **h** $-9 < z < -3$
2 $x > 1$ **3** 1, 2, 3, 4, 5
4 7 **5** 4, 5, 6
6 b i $-3 < x < 3$ **ii** $-5 < x < 5$
iii $x > +7$ or $x < -7$
7 a $-1 < x < 1$ **b** $x > 4, x < -4$
c $-3 \leqslant x \leqslant 3$ **d** $-4 < x < 4$
e $x > 5, x < -5$ **f** $x \geqslant 4, x \leqslant -4$

8 a $x \geqslant 6$ or $x \leqslant -2$ **b** $-4 < x < -2$
c $x \geqslant 6$ or $x \leqslant -1$ **d** $-3 < x < -2$
e $x \geqslant 4$ or $x \leqslant 3$ **f** $1 \leqslant x \leqslant 2$
g $\frac{3}{2} < x < 4$ **h** $-\frac{2}{3} < x < 5$
i $-4\frac{1}{2} < x < \frac{1}{2}$ **j** $x \geqslant 3$ or $x \leqslant \frac{4}{3}$
k $x \leqslant -\frac{1}{2}, x \geqslant \frac{3}{2}$ **l** $-2 < x < \frac{13}{3}$
m $2 < x < \frac{5}{2}$ **n** $x > \frac{5}{2}$ or $x < \frac{1}{3}$
9 $x > 4$ **10** $x > 5$ **11** $x < -3$
12 $p \leqslant 4$ and $p \geqslant 20$ **13** $-1 < x < 3$ and $x > 3$

Algebraic Fractions

Example 1

Simplify

a $\dfrac{12}{15}$ **b** $\dfrac{7a}{5a^2}$ **c** $\dfrac{3x+9}{6x}$

d $\dfrac{x^2+2x}{x+2}$ **e** $\dfrac{x^2+2x-3}{x^2+5x+6}$

a $\dfrac{12}{15}$. Divide numerator and denominator by 3, we have $\dfrac{12}{15} = \dfrac{4}{5}$.

b $\dfrac{7a}{5a^2}$. Dividing numerator and denominator by a, we have $\dfrac{7a}{5a^2} = \dfrac{7}{5a}$

c $\dfrac{3x+9}{6x}$. A common mistake is often made with fractions like this
Do **not** cancel the xs.

Instead we write $\dfrac{3x+9}{6x} = \dfrac{3(x+3)}{6x} = \dfrac{(x+3)}{2x}$.

d Factorising the numerator, $\dfrac{x^2+2x}{x+2} = \dfrac{x(x+2)}{x+2} = x$

e Factorising $\dfrac{x^2+2x-3}{x^2+5x+6} = \dfrac{(x-1)(x+3)}{(x+2)(x+3)} = \dfrac{x-1}{x+2}$

Example 2

a Write as a single fraction $\dfrac{2}{x} + \dfrac{3}{y}$

The L.C.M. of x and y is xy.

$$\therefore \frac{2}{x} + \frac{3}{y} = \frac{2y}{xy} + \frac{3x}{xy} = \frac{2y+3x}{xy}$$

b Write as a single fraction $\dfrac{4}{x} + \dfrac{5}{x-1}$

The L.C.M. of x and $(x-1)$ is $x(x-1)$

$$\therefore \frac{4}{x} + \frac{5}{x-1} = \frac{4(x-1)+5x}{x(x-1)} = \frac{9x-4}{x(x-1)}$$

Extra Support

For further explanations on this topic please refer to the following:

Algebraic Fractions Adding - https://corbettmaths.com/2013/01/19/adding-algebraic-fractions/

Algebraic Fractions – Division - https://corbettmaths.com/2013/01/19/dividing-algebraic-fractions/

Algebraic Fractions – Multiplication https://corbettmaths.com/2013/01/19/multiplying-algebraic-fractions/

Exercise 1A

1 Simplify:

a $\frac{4x+4}{x+1}$ **b** $\frac{2x-1}{6x-3}$ **c** $\frac{x+4}{x+2}$

d $\frac{x+\frac{1}{2}}{4x+2}$ **e** $\frac{4x+2y}{6x+3y}$ **f** $\frac{a+3}{a+6}$

g $\frac{5p-5q}{10p-10q}$ **h** $\frac{\frac{1}{2}a+b}{2a+4b}$ **i** $\frac{x^2}{x^2+3x}$

j $\frac{x^2-3x}{x^2-9}$ **k** $\frac{x^2+5x+4}{x^2+8x+16}$ **l** $\frac{x^3-2x^2}{x^2-4}$

m $\frac{x^2-4}{x^2+4}$ **n** $\frac{x+2}{x^2+5x+6}$ **o** $\frac{2x^2-5x-3}{2x^2-7x-4}$

p $\frac{\frac{1}{2}x^2+x-4}{\frac{1}{4}x^2+\frac{3}{2}x+2}$ **q** $\frac{3x^2-x-2}{\frac{1}{2}x+\frac{1}{3}}$ **r** $\frac{x^2-5x-6}{\frac{1}{3}x-2}$

Answers

Exercise 1A

1 a 4 **b** $\frac{1}{3}$ **c** $\frac{x+4}{x+2}$ **d** $\frac{1}{4}$

e $\frac{2}{3}$ **f** $\frac{a+3}{a+6}$ **g** $\frac{1}{2}$ **h** $\frac{1}{4}$

i $\frac{x}{x+3}$ **j** $\frac{x}{x+3}$ **k** $\frac{x+1}{x+4}$ **l** $\frac{x^2}{x+2}$

m $\frac{x^2-4}{x^2+4}$ **n** $\frac{1}{x+3}$ **o** $\frac{x-3}{x-4}$ **p** $\frac{2(x-2)}{x+2}$

q $6(x-1)$ **r** $3(x+1)$

Exercise 1B

1 Simplify:

a $\frac{a}{d} \times \frac{a}{c}$ **b** $\frac{a^2}{c} \times \frac{c}{a}$ **c** $\frac{2}{x} \times \frac{x}{4}$

d $\frac{3}{x} \div \frac{6}{x}$ **e** $\frac{4}{xy} \div \frac{x}{y}$ **f** $\frac{2r^2}{5} \div \frac{4}{r^3}$

g $(x+2) \times \frac{1}{x^2-4}$ **h** $\frac{1}{a^2+6a+9} \times \frac{a^2-9}{2}$

i $\frac{x^2-3x}{y^2+y} \times \frac{y+1}{x}$ **j** $\frac{y}{y+3} \div \frac{y^2}{y^2+4y+3}$

k $\frac{x^2}{3} \div \frac{2x^3-6x^2}{x^2-3x}$ **l** $\frac{4x^2-25}{4x-10} \div \frac{2x+5}{8}$

m $\frac{x+3}{x^2+10x+25} \times \frac{x^2+5x}{x^2+3x}$ **n** $\frac{3y^2+4y-4}{10} \div \frac{3y+6}{15}$

o $\frac{x^2+2xy+y^2}{2} \times \frac{4}{(x-y)^2}$

Answers

Exercise 1B

1 a $\frac{a^2}{cd}$ **b** a **c** $\frac{1}{2}$ **d** $\frac{1}{2}$

e $\frac{4}{x^2}$ **f** $\frac{r^5}{10}$ **g** $\frac{1}{x-2}$ **h** $\frac{a-3}{2(a+3)}$

i $\frac{x-3}{y}$ **j** $\frac{y+1}{y}$ **k** $\frac{x}{6}$ **l** 4

m $\frac{1}{x+5}$ **n** $\frac{3y-2}{2}$ **o** $\frac{2(x+y)^2}{(x-y)^2}$

Exercise 1C

1 Simplify:

a $\frac{1}{p} + \frac{1}{q}$ **b** $\frac{a}{b} - 1$ **c** $\frac{1}{2x} + \frac{1}{x}$

d $\frac{3}{x^2} - \frac{1}{x}$ **e** $\frac{3}{4x} + \frac{1}{8x}$ **f** $\frac{x}{y} + \frac{y}{x}$

g $\frac{1}{x+2} - \frac{1}{x+1}$ **h** $\frac{2}{x+3} - \frac{1}{x-2}$ **i** $\frac{1}{3}(x+2) - \frac{1}{2}(x+3)$

j $\frac{3x}{(x+4)^2} - \frac{1}{(x+4)}$ **k** $\frac{1}{2(x+3)} + \frac{1}{3(x-1)}$

l $\frac{2}{x^2+2x+1} + \frac{1}{x+1}$ **m** $\frac{3}{x^2+3x+2} - \frac{2}{x^2+4x+4}$

n $\frac{2}{a^2+6a+9} - \frac{3}{a^2+4a+3}$ **o** $\frac{2}{y^2-x^2} + \frac{3}{y-x}$

p $\frac{x+2}{x^2-x-12} - \frac{x+1}{x^2+5x+6}$ **q** $\frac{3x+1}{(x+2)^3} - \frac{2}{(x+2)^2} + \frac{4}{(x+2)}$

Answers

Exercise 1C

1 **a** $\dfrac{q+p}{pq}$ **b** $\dfrac{a-b}{b}$ **c** $\dfrac{3}{2x}$

d $\dfrac{3-x}{x^2}$ **e** $\dfrac{7}{8x}$ **f** $\dfrac{x^2+y^2}{xy}$

g $-\dfrac{1}{(x+2)(x+1)}$ **h** $\dfrac{x-7}{(x+3)(x-2)}$ **i** $\dfrac{-x-5}{6}$

j $\dfrac{2x-4}{(x+4)^2}$ **k** $\dfrac{5x+3}{6(x+3)(x-1)}$

l $\dfrac{x+3}{(x+1)^2}$ **m** $\dfrac{x+4}{(x+1)(x+2)^2}$

n $\dfrac{-a-7}{(a+1)(a+3)^2}$ **o** $\dfrac{2+3y+3x}{(y+x)(y-x)}$

p $\dfrac{7x+8}{(x+2)(x+3)(x-4)}$ **q** $\dfrac{4x^2+17x+13}{(x+2)^2}$

Coordinate Geometry

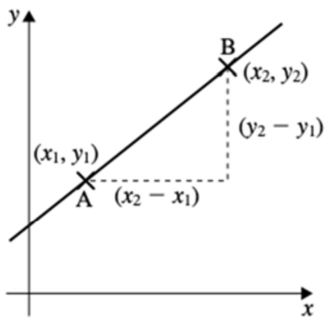
Gradient

The gradient of a straight line is defined as

$$\frac{\text{the increase in } y}{\text{the increase in } x}$$

In the diagram, the gradient of the line AB is

$$\left(\frac{y_2-y_1}{x_2-x_1}\right) \quad \text{or} \quad \left(\frac{y_1-y_2}{x_1-x_2}\right)$$



Length of a straight line

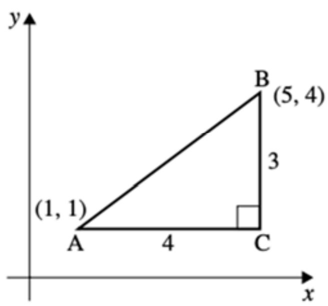
Find the length of the line AB joining A(1, 1) and B(5, 4).

In triangle ACB, AC = 4 and BC = 3.

By Pythagoras' theorem, $AB^2 = AC^2 + BC^2$

$AB^2 = 4^2 + 3^2$

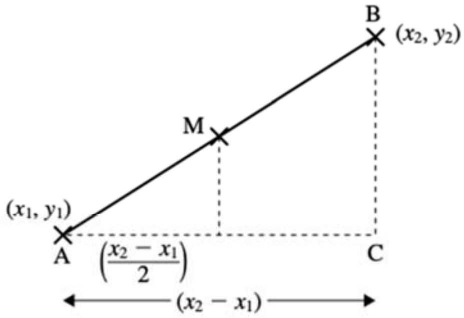
$AB = 5$ units.



In general the length of the line joining (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Mid-point of a line



M is the mid-point of AB.

length AC $= x_2 - x_1$

The x coordinate of M is

$$x_1 + \left(\frac{x_2 - x_1}{2}\right) = \left(\frac{x_1 + x_2}{2}\right)$$

Similarly the y coordinate of M is

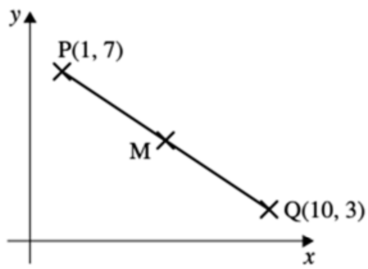
$$\left(\frac{y_1 + y_2}{2}\right)$$

So the mid-point has coordinates

$$\left[\left(\frac{x_1 + x_2}{2}\right), \left(\frac{y_1 + y_2}{2}\right)\right]$$

Example 1

In the diagram,



a gradient of PQ $= \dfrac{7 - 3}{1 - 10} = \dfrac{4}{-9} = -\dfrac{4}{9}$.

b coordinates of mid-point, M, are

$$\left(\frac{1 + 10}{2}, \frac{7 + 3}{2}\right)$$

i.e. $(\frac{11}{2}, 5)$

c length of line PQ $= \sqrt{(1 - 10)^2 + (7 - 3)^2}$

$= \sqrt{81 + 16}$

$= \sqrt{97}$ units

Example 2

a Write down the gradient of the line joining the points $(k, k + 1)$ and $(4, 7)$.

b Find the value of k if the gradient of the line is 2.

c Find the value of k if the line is parallel to the x axis.

a Gradient of line is $\dfrac{k + 1 - 7}{k - 4} = \dfrac{k - 6}{k - 4}$

b Gradient of line $= 2$

$$\therefore \quad \frac{k - 6}{k - 4} = 2$$

$$k - 6 = 2k - 8$$

$$2 = k$$

c If the line is parallel to the x axis, its gradient is zero.

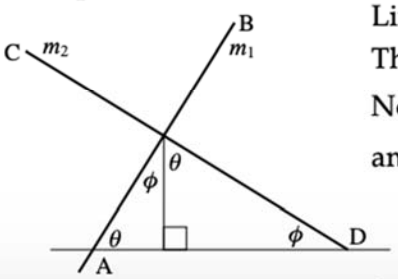
$$\therefore \quad \frac{k - 6}{k - 4} = 0$$

$$k = 6.$$

Parallel and perpendicular lines

a When two lines are parallel they have the same gradient.

b Perpendicular lines



Lines AB and CD are perpendicular.

The gradients of AB and CD are m_1 and m_2 respectively.

Now $\quad m_1 = \tan\theta$

and $\quad m_2 = -\tan\phi$

$$= -\frac{1}{\tan\theta}$$

$$\therefore \quad m_1 \times m_2 = \tan\theta\left(-\frac{1}{\tan\theta}\right)$$

Example 3

Line PQ has gradient $\frac{1}{2}$.

a Find the gradient of line RS which is parallel to PQ.

b Find the gradient of line TU which is perpendicular to PQ.

a Parallel lines have the same gradient. The gradient of line RS is $\frac{1}{2}$.

b Let the gradient of line TU be m.

We have $m \times \frac{1}{2} = -1$

$\therefore \quad m = -2$

The gradient of line TU is -2.

Extra Support

The equation of a straight line - https://corbettmaths.com/2013/05/29/finding-the-equation-of-a-straight-line/

Equation of a line through two points - https://corbettmaths.com/2013/05/29/finding-the-equation-passing-through-two-points/

Parallel lines - https://corbettmaths.com/2013/06/06/graphs-parallel-lines/

Perpendicular lines - https://corbettmaths.com/2013/06/06/perpendicular-lines-2/

Mid-point of a line - https://corbettmaths.com/2013/04/15/midpoint-of-a-line/

Exercise 2A

1 Find the gradient of the straight line which passes through the following pairs of points:

a (1, 3) and (2, 5) **b** (3, 7) and (5, 11)

c (2, 8) and (5, 20) **d** (4, 1) and (1, 16)

e (−1, 2) and (−3, −8) **f** (−1, −2) and (−4, −20)

g (−1, −3) and (−5, 9) **h** (2, −2) and (5, −11)

i (1, −3) and (2, −3) **j** (1, 7) and (1, 9)

2 **a** Write down the gradient of the line joining the points (a, $2a$) and (3, 5).

b Find the value of a if the gradient of the line is 3.

c Find the value of a if the line is parallel to the x axis.

3 **a** Write down the gradient of the line joining the points ($2m$, n) and (3, −4).

b Find the value of n if the line is parallel to the x axis.

c Find the value of m if the line is parallel to the y axis.

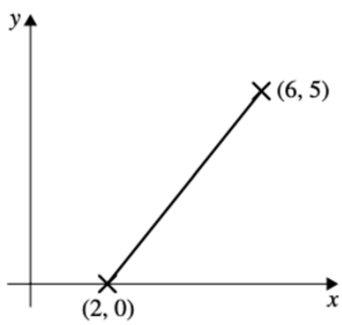
4 Find the midpoint of the following pairs of points:

a (1, 4) and (1, 6) **b** (3, 7) and (5, 7) **c** (3, 9) and (7, 19)

d (5, −1) and (3, −7) **e** (−1, 1) and (7, −11) **f** (−9, −1) and (11, −13)

5 The midpoint of the line joining (a, b) and (2, 4) has coordinates (3, −1). Find the values of a and b.

6 Find the length of the line joining the points (2, 0) and (6, 5).



7 Find the length of the straight line joining the following pairs of points.

a (4, 2) and (1, 0) **b** (6, 3) and (2, −1) **c** (−1, 0) and (4, −2)

Answers

EXERCISE 2A *page 54*

1 a 2 b 2 c 4 d −5
e 5 f 6 g −3 h −3
i 0 j infinite

2 a $\frac{2a-5}{a-3}$ b 4 c $\frac{5}{2}$

3 a $\frac{n+4}{2m-3}$ b −4 c $\frac{3}{2}$

4 a (1, 5) b (4, 7) c (5, 14)
d (4, −4) e (3, −5) f (1, −7)

5 $a = 4, b = -6$ 6 $\sqrt{41}$

7 a $\sqrt{13}$ b $4\sqrt{2}$ c $\sqrt{29}$

8 0, 7 9 $-\frac{1}{3}$

10 $5, -\frac{1}{2}, -\frac{2}{3}, \frac{1}{5}, -\frac{4}{3}$ 11 −1

12 b (6, 0) c $(4\frac{1}{2}, 3\frac{1}{2})$ d 29

13 3 14 a $4, \frac{32}{3}$ c $(1\frac{1}{2}, 2\frac{1}{2})$

EXERCISE 2B *page 57*

2 a parallel b parallel
c perpendicular d perpendicular
e perpendicular f neither
g perpendicular

3 A $y = 2x - 3$ B $y = \frac{1}{3}x + 2$ C $x + y = 6$

4 a $x - 3y + 7 = 0$ b $2x - y = 0$
c $5x - y - 9 = 0$ d $2x - 3y - 10 = 0$

5 a $y = x + 1$ b $y = 2x + 3$
c $y = 5x - 1$ d $y = 7x - 2$
e $y = -3x - 4$ f $y = -2x + 11$

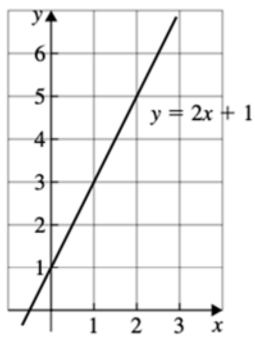
6 a $y = 5x - 1$ b $y = -4x - 5$
c $3x - 4y + 11 = 0$ d $2x - 5y + 19 = 0$
e $y = 3x - 1$ f $5x - 2y + 26 = 0$

7 a 3 b $-\frac{1}{3}$ c 4 d $-\frac{3}{2}$

8 a $5x + y - 9 = 0$ b $x + 3y - 14 = 0$
c $3x + 4y - 34 = 0$ d $3x + y - 17 = 0$
e $5x + 3y + 2 = 0$ f $3x + 4y - 23 = 0$
g $2x + 3y - 21 = 0$

The equation of a straight line

- The line $y = mx + c$ has gradient m and c is the y intercept.
  - **a** In the diagram the line $y = 2x + 1$ is shown. The gradient is 2 and the y intercept is 1.
  - **b** Similarly the line $y = -4x + 11$ has gradient -4 and y intercept 11.



Example 5

Find the equation of the line which passes through (3, 11) and (−1, 27).

First find the gradient.

$$\text{Gradient} = \frac{27 - 11}{-1 - 3} = -4$$

Now there are two methods to find the equation of the line.

Method A [using $y = mx + c$]

The gradient is -4 so the equation is $y = -4x + c$.

The line passes through the point (3, 11), so replace x with 3 and y with 11.

$$11 = -4 \times 3 + c$$

$$c = 23$$

So the equation of the line is $y = -4x + 23$.

Method B [using $y - y_1 = m(x - x_1)$]

Choose (3, 11) for (x_1, y_1)

Using the form $\quad y - y_1 = m(x - x_1)$

we have $\quad y - 11 = -4(x - 3)$

so $\quad y = -4x + 23$

Notice that we obtain the same equation if we choose (−1, 27) as (x_1, y_1). You should do this for yourself.

Example 6

Find the equation of straight line which passes through (1, 4) and which is parallel to $2y + 3x = 7$.

The line $2y + 3x = 7$ may be written as

$$2y = -3x + 7$$

$$y = -\tfrac{3}{2}x + \tfrac{7}{2}$$

The gradient of the line is $-\frac{3}{2}$.

The line required also has gradient $-\frac{3}{2}$ and passes through (1, 4).

The equation of the line is $y - 4 = -\frac{3}{2}(x - 1)$

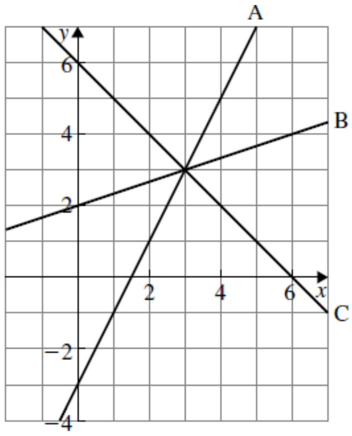
$$2y - 8 = -3x + 3$$

$$3x + 2y = 11$$

or $3x + 2y - 11 = 0$ [This is in the form '$ax + by + c = 0$'.]

Exercise 2B

3 Find the equation of lines A, B and C.



4 Find the equation of the straight line (in the form $ax + by + c = 0$ where a, b and c are integers) which:

a Passes through (2, 3) and is parallel to $3y = x - 1$.

b Passes through (−1, −2) and is parallel to $y = 2x + 3$.

c Passes through the mid-point of (1, 3) and (5, 9) and has gradient 5.

d Passes through the mid-point of (−5, 2) and (3, −10) and has gradient $\frac{2}{3}$.

5 Find the equation of the straight line which passes through the following pairs of points:

a (2, 3) and (3, 4)

b (1, 5) and (3, 9)

c (3, 14) and (7, 34)

d (−1, −9) and (3, 19)

e (−1, −1) and (3, −13)

f (2, 7) and (5, 1)

6 Find the equation of the straight line which:

a Passes through (1, 4) and has gradient 5 (in the form $y = mx + c$).

b Passes through (−2, 3) and has gradient −4 (in the form $y = mx + c$).

c Passes through (3, 5) and has gradient $\frac{3}{4}$ (in the form $ax + by + c = 0$).

d Passes through (−2, 3) and has gradient $\frac{2}{5}$ (in the form $ax + by + c = 0$).

e Passes through (1, 2) and is parallel to $y = 3x + 2$ (in the form $y = mx + c$).

f Passes through (−4, 3) and is parallel to $2y = 5x - 1$ (in the form $ax + by + c = 0$).

7 Write down the gradients of the following lines:

a $2y = 6x - 11$ b $3y + x = 12$ c $4x - y = 4$ d $3x + 2y + 10 = 0$

8 Find the equation of the straight line (in the form $ax + by + c = 0$ where a, b and c are integers) which:

a Passes through (1, 4) and is perpendicular to $5y = x + 2$.

b Passes through (−1, 5) and is perpendicular to $y = 3x + 1$.

c Passes through (2, 7) and is perpendicular to $3y = 4x + 5$.

d Passes through the midpoint of (2, 5) and (8, −1) and is perpendicular to $3y = x + 1$.

e Passes through the midpoint of (−7, 4) and (5, −2) and is perpendicular to a line with gradient $\frac{3}{5}$.

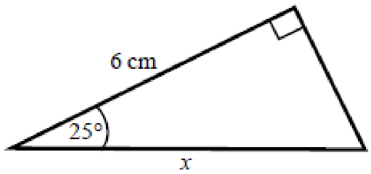
f Is the perpendicular bisector of the line through (4, 9) and (−2, 1).

g Is the perpendicular bisector of the line through (4, 13) and (−4, 1).

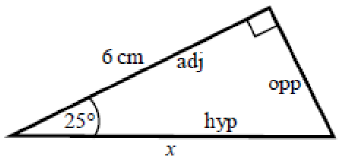
Answers – on page 31

Trigonometry

Example 1 Calculate the length of side *x*.
Give your answer correct to 3 significant figures.



Solution



$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

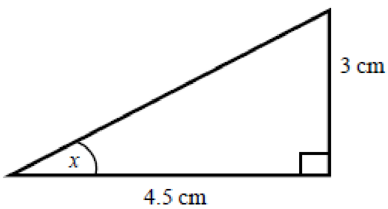
$$\cos 25° = \frac{6}{x}$$

$$x = \frac{6}{\cos 25°}$$

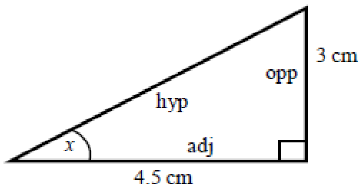
x = 6.620 267 5...

x = 6.62 cm

Example 2 Calculate the size of angle *x*.
Give your answer correct to 3 significant figures.



Solution



$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{3}{4.5}$$

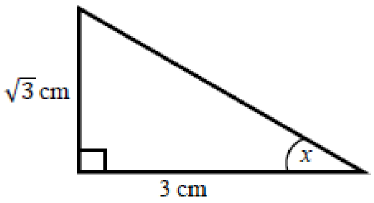
$$x = \tan^{-1}\left(\frac{3}{4.5}\right)$$

x = 33.690 067 5...

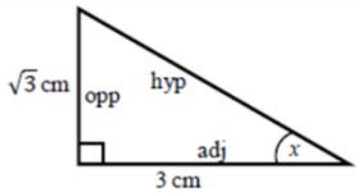
x = 33.7°

Example 3 Calculate the exact size of angle x.

REMEMBER EXACT MEANS NO DECIMALS



Solution



$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{\sqrt{3}}{3}$$

x = 30°

Extra Support

Trigonometry Introduction - https://corbettmaths.com/2013/03/30/trigonometry-introduction/

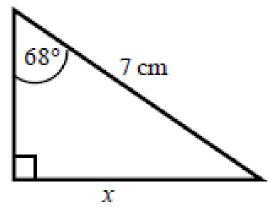
Trigonometry finding Sides - https://corbettmaths.com/2013/03/30/trigonometry-missing-sides/

Trigonometry finding Angles- https://corbettmaths.com/2013/03/30/trigonometry-missing-angles/

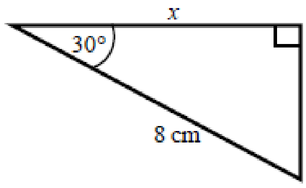
Practice

1 Calculate the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

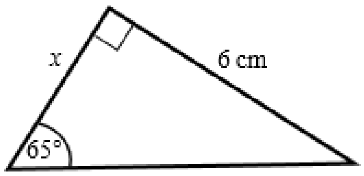
a



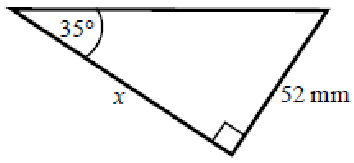
b



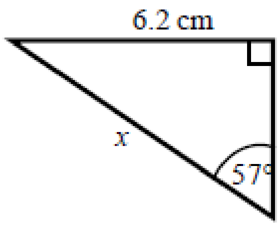
c



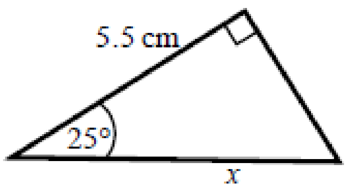
d



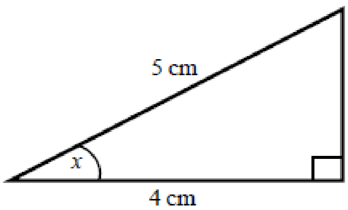
e

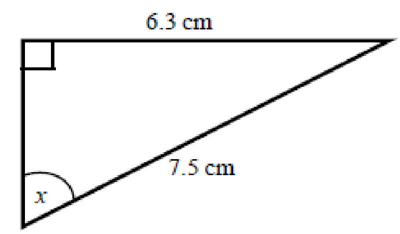


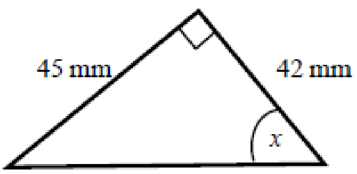
f

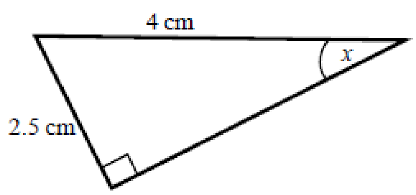


2 Calculate the size of angle x in each triangle.
Give your answers correct to 1 decimal place.

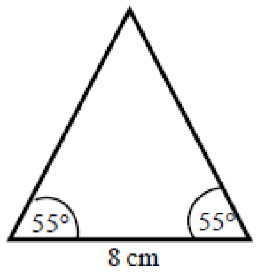




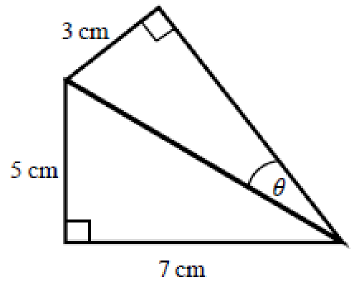




3 Work out the height of the isosceles triangle.
Give your answer correct to 3 significant figures.

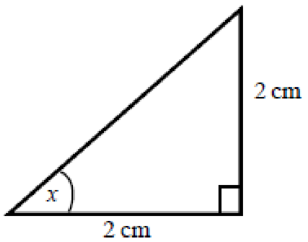


4 Calculate the size of angle θ.
Give your answer correct to 1 decimal place.

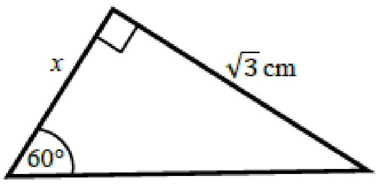


5 Find the exact value of x in each triangle.

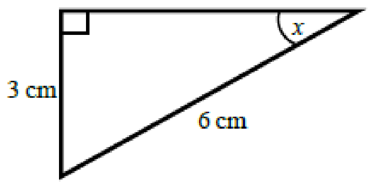
a



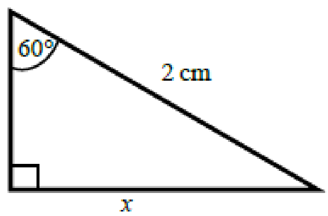
b



c



d



<u>Answers</u>

1 **a** 6.49 cm **b** 6.93 cm **c** 2.80 cm
d 74.3 mm **e** 7.39 cm **f** 6.07 cm

2 **a** 36.9° **b** 57.1° **c** 47.0° **d** 38.7°

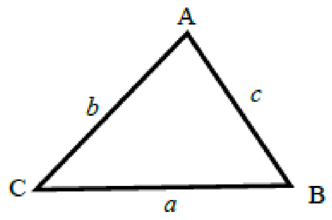
3 5.71 cm

4 20.4°

5 **a** 45° **b** 1 cm **c** 30° **d** $\sqrt{3}$ cm

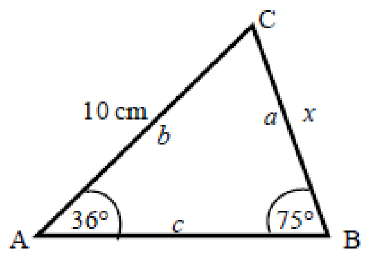
Sine Rule

- *a* is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.

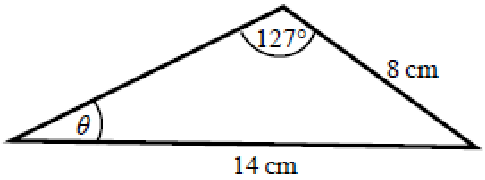


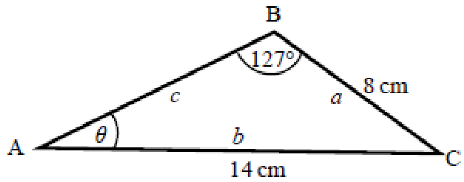
- You can use the sine rule to find the length of a side when its opposite angle and another opposite side and angle are given.
- To calculate an unknown side use the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- Alternatively, you can use the sine rule to find an unknown angle if the opposite side and another opposite side and angle are given.
- To calculate an unknown angle use the formula $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Example Work out the length of side *x*.
Give your answer correct to 3 significant figures.

	**1** Always start by labelling the angles and sides.
$\frac{a}{\sin A} = \frac{b}{\sin B}$	**2** Write the sine rule to find the side.
$\frac{x}{\sin 36°} = \frac{10}{\sin 75°}$	**3** Substitute the values *a*, *b*, *A* and *B* into the formula.
$x = \frac{10 \times \sin 36°}{\sin 75°}$	**4** Rearrange to make *x* the subject.
x = 6.09 cm	**5** Round your answer to 3 significant figures and write the units in your answer.

Example 7 Work out the size of angle ϑ.
Give your answer correct to 1 decimal pla



<table>
<tr>
<td>


$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{8} = \frac{\sin 127°}{14}$$

$$\sin \theta = \frac{8 \times \sin 127°}{14}$$

$\vartheta = 27.2°$
</td>
<td>
1 Always start by labelling the angles and sides.

2 Write the sine rule to find the angle.

3 Substitute the values a, b, A and B into the formula.

4 Rearrange to make sin ϑ the subject.

5 Use $\sin^{-1}$ to find the angle. Round your answer to 1 decimal place and write the units in your answer.
</td>
</tr>
</table>

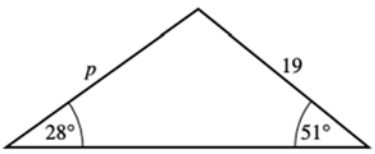
Extra Support

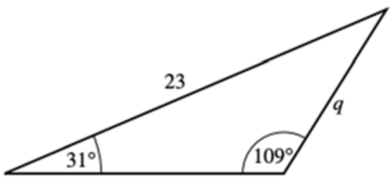
The Sine Rule Sides - https://corbettmaths.com/2013/05/03/sine-rule-missing-sides/

The Sine Rule Angles - https://corbettmaths.com/2019/04/24/sine-rule-angles/

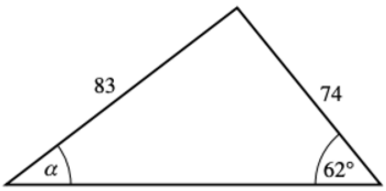
Exercise 9A

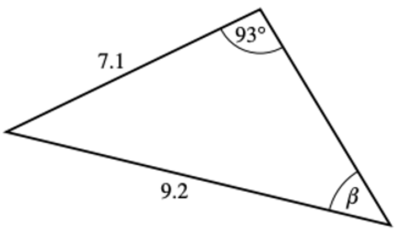
1 Show that $p = 31.5$ (to 3 significant figures) and $q = 12.5$ (to 3 significant figures) in the following:



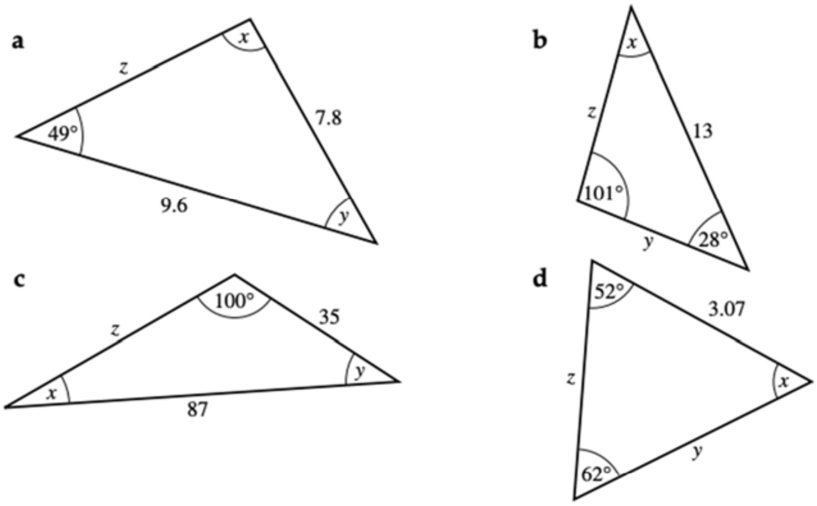


2 Show that $\alpha = 51.9°$ and $\beta = 50.4°$ (to 1 decimal place) in the following triangles:





3 Find x, y and z in the following triangles (give angles 1 decimal place) and lengths to 3 significant figures) – calculate these in alphabetical order



4 In $\triangle ABC$, $\hat{A} = 62°$, BC = 8, AB = 7. Find $\hat{C}$.

5 In $\triangle XYZ$, $\hat{Y} = 97.3°$, XZ = 22, XY = 14. Find $\hat{Z}$.

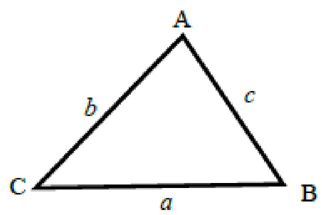
6 In $\triangle PQR$, $\hat{Q} = 100°$, $\hat{R} = 21°$, PQ = 3.1. Find PR.

Answers

3 a $x = 68.3°,\ y = 62.7°,\ z = 9.18$
b $x = 51.0°,\ y = 10.3°,\ z = 6.22$
c $x = 23.3°,\ y = 56.7°,\ z = 73.8$
d $x = 66.0°,\ y = 2.74,\ z = 3.18$
4 50.6° **5** 39.1° **6** 8.52

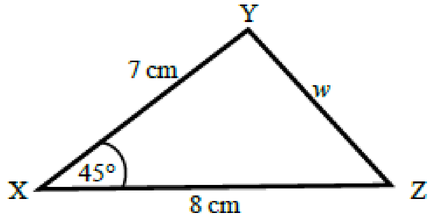
Cosine Rule

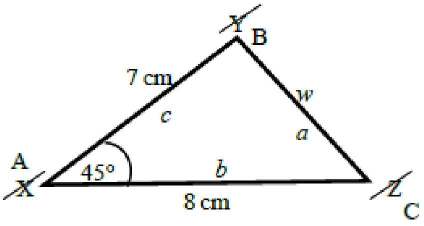
- *a* is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.



- You can use the cosine rule to find the length of a side when two sides and the included angle are given.
- To calculate an unknown side use the formula $a^2 = b^2 + c^2 - 2bc\cos A$.
- Alternatively, you can use the cosine rule to find an unknown angle if the lengths of all three sides are given.
- To calculate an unknown angle use the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Example Work out the length of side w.
Give your answer correct to 3 s f.





$a^2 = b^2 + c^2 - 2bc\cos A$

$w^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 45°$

$w^2 = 33.804\,040\,51...$

$w = \sqrt{33.80404051}$

$w = 5.81$ cm

1 Always start by labelling the angles and sides.

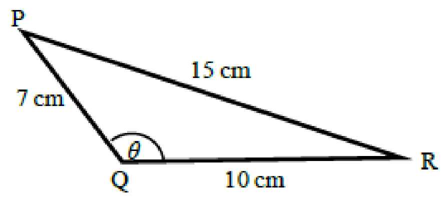
2 Write the cosine rule to find the side.

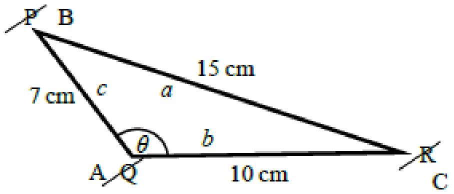
3 Substitute the values a, b and A into the formula.

4 Use a calculator to find w^2 and then w.

5 Round your final answer to 3 significant figures and write the units in your answer.

Example Work out the size of angle ϑ.
Give your answer correct to 1 dp





$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\theta = \dfrac{10^2 + 7^2 - 15^2}{2 \times 10 \times 7}$

$\cos\theta = \dfrac{-76}{140}$

$\vartheta = 122.878\,349...$

$\vartheta = 122.9°$

1 Always start by labelling the angles and sides.

2 Write the cosine rule to find the angle.

3 Substitute the values a, b and c into the formula.

4 Use $\cos^{-1}$ to find the angle.

5 Use your calculator to work out $\cos^{-1}(-76 \div 140)$.

6 Round your answer to 1 decimal place and write the units in your answer.

Extra Support

Cosine Rule Sides - Cosine Rule – Missing Sides Video – Corbettmaths

Cosine Rule Angles - https://corbettmaths.com/2013/04/04/cosine-rule-missing-angles/

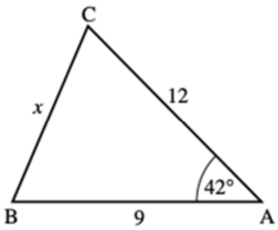
Exercise 9B

EXERCISE 9B

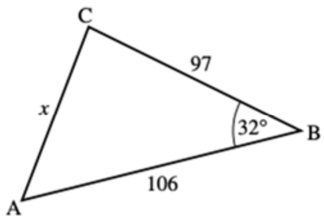
All lengths are in cm unless otherwise stated.

1 Find x (to 3 significant figures) in the following triangles:

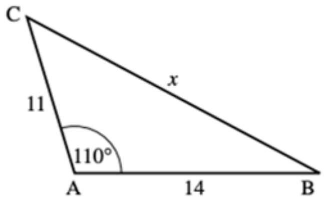
a



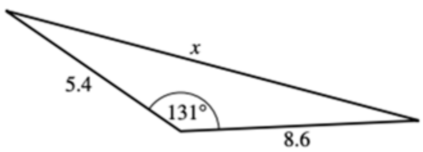
b



c

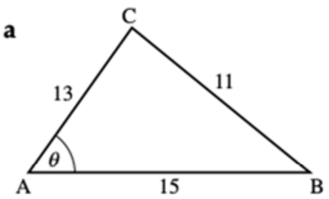


d

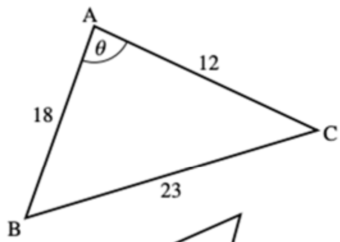


2 Find θ (to 1 decimal place) in the triangles shown below:

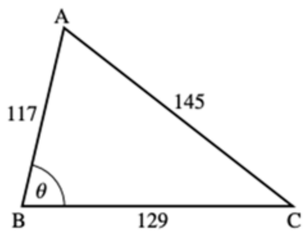
a



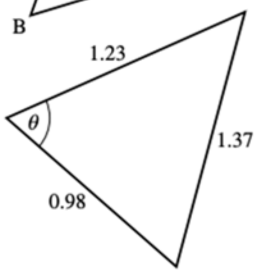
b



c



d



3 A man walks 300 m from a point P in a straight line and then turns through an angle θ and walks 120 m in a straight line. If he ends up 340 m from P, find θ (to 1 decimal place).

4 In $\triangle$LMN, LM = 5.3 cm, MN = 7.9 cm, $\hat{M}$ = 127°. Find LN.

5 In a triangle ABC, AB = 3.8 cm, BC = 5.1 cm and the angle at B is 40°. Find the length of AC (to 3 significant figures).

6 A triangle XYZ is such that XY = 19 mm and YZ = 23 mm. If XZ = 35 mm, find the angle at Y (to 1 decimal place).

7 A man walks 5 km due North and then a further 4 km on a bearing of 120°.

a Draw a clear sketch of this journey, stating all known lengths and angles.

b How far is he from where he started (to 3 significant figures)?

8 A rally car leaves its base on a bearing of 030° and drives in a desert for 12 km. It then turns and drives a further 8 km on a bearing of 100° after which it breaks down.

a Draw a clear sketch of this journey, stating all known lengths and angles.

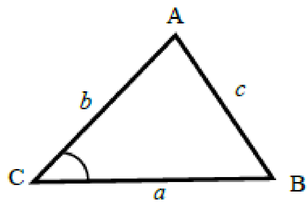
b A recovery vehicle leaves base to pick up the car. How far must it travel (to 3 significant figures) and on what bearing should it head (to the nearest degree)?

Answers

1 a 8.03 cm **b** 56.6 cm
c 20.6 cm **d** 12.8 cm
2 a 45.6° **b** 98.1°
c 72.0° **d** 75.7°
3 98.9° (or 81.1°) **4** 11.9 cm
5 3.28 cm **6** 112.5°
7 4.58 km **8** 16.5 km, 057°
9 c 4

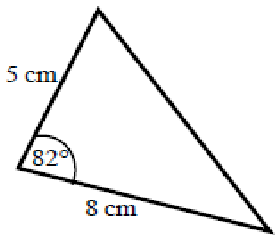
Area of a Triangle

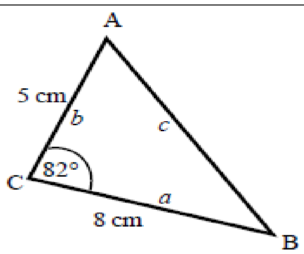
- a is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.
- The area of the triangle is $\frac{1}{2}ab\sin C$.



Example

Find the area of the triangle.



	**1** Always start by labelling the sides and angles of the triangle.
Area = $\frac{1}{2}ab\sin C$	**2** State the formula for the area of a triangle.
Area = $\frac{1}{2}\times 8\times 5\times \sin 82°$	**3** Substitute the values of a, b and C into the formula for the area of a triangle.
Area = 19.805 361...	**4** Use a calculator to find the area.
Area = 19.8 cm^2	**5** Round your answer to 3 significant figures and write the units in your answer.

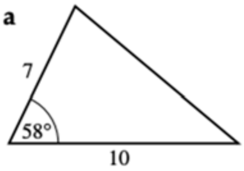
Extra Support

Area of a triangle video - https://corbettmaths.com/2012/08/02/area-of-a-triangle-sinetrigonometry/

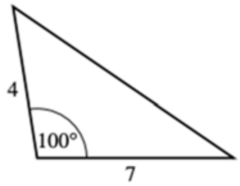
Exercise 9D

1 Find the area of each triangle. All lengths are in cm.

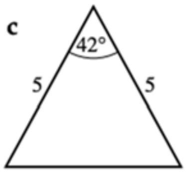
a



b



c



2 Find the area of an equilateral triangle of side 10 cm.

3 Find the area of a parallelogram ABCD with AB = 8 cm, AD = 5 cm and $B\hat{A}D = 74°$.

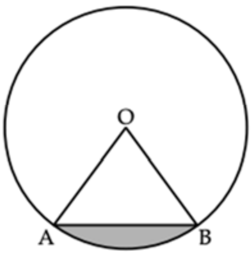
4 Find the area of a regular hexagon of side 8 cm.

5 The area of an equilateral triangle is 50 cm^2. Find the length of a side of the triangle.

6 Find the length of a side of an equilateral triangle of area 212 m^2.

7 A rhombus has an area of 60 cm^2 and adjacent angles of 40° and 140°. Find the length of a side of the rhombus.

8 In the diagram, O is the centre of a circle of radius 5 cm and $A\hat{O}B = 88°$. Find the shaded area.



9* The area of a regular pentagon is 150 cm^2. Calculate the length of one side of the pentagon.

Answers

1 a 29.7 cm^2 **b** 13.8 cm^2 **c** 8.36 cm^2
2 43.3 cm^2 **3** 38.5 cm^2 **4** 166 cm^2
5 10.7 cm **6** 22.1 cm **7** 9.66 cm
8 6.71 cm^2 **9** 9.34 cm

GCSE to A Level Diagnostic Test 2025

Answer all these questions, showing your working out clearly. If you need more space, use a separate sheet of paper. Then mark your work and bring this sheet to your first lesson.

1. Simplify

   a) $x^{-2} \times x^6$ b) $\left(2a^2b^3\right)^5$ c) $\frac{12x^4y^2z + (3x)^3}{(6xy)^2}$

2. Solve the following equations:

a) $27^{n-1} = 3^6$ b) $x^{-\frac{2}{3}} = 4$

3. Simplify a) $\sqrt{32} - 3\sqrt{2}$ b) $2\sqrt{6} \times 5\sqrt{15}$

   Rationalise c) $\frac{20}{\sqrt{5}}$ d) $\frac{11}{\sqrt{7}+2}$ e) $\frac{2+3\sqrt{3}}{2-3\sqrt{3}}$

4. Expand and simplify a) $(4x+1)(7-x)$ b) $\left(3+\sqrt{5}\right)^2$

   Fully factorise c) $12a^5 - 3a^2$ d) $4x^2 - 25$

   Find a, b and c in the following: e) $3x^3 - 8x^2 + 11x - 14 = (x-2)(ax^2 + bx + c)$

5. Solve the following equations by factorising first:

   a) $x^2 + 2x = 35$ b) $x^2 - 7x = 0$ c) $6x^2 + 7x - 3 = 0$

6. Use the quadratic formula to solve $2x^2 - 6x - 5 = 0$ giving your solutions in simplified surd form

7. a) Find the minimum value of $x^2 + 6x - 11$

Complete the square for the following functions

b) $y = 2x^2 - 8x + 23$

c) $y = -x^2 - 7x + 15$

8. Solve these simultaneous equations:

a) $3x - y = 11$ and $x + y = 5$

b) $x + 3y = 11$ and $5x - y = 7$

c) $xy = 15$ and $x + y = 8$

9. Solve the inequalities:

a) $4x - 7 > 11$

b) $26 \le 3x - 1 \le 55$

c) $6x^2 + 5x - 4 > 0$

10. a) Calculate $\frac{3b}{8} + \frac{1}{2}b$

b) Simplify $\frac{1}{3x+2} + \frac{2}{x-5}$

c) Find a solution of $\frac{1}{x-1} + \frac{1}{x} = \frac{7}{12}$

11. For the two points A (1,1) and B(–3, 4),

a) Calculate the distance between A and B

b) Find the gradient of the line that passes through A and B

c) Find the mid-point of AB

12. From the following 6 equations of lines, find one pair that are a) parallel and b) perpendicular

i) $y = \frac{2}{3}x - 7$ ii) $y = -3x + 7$ iii) $6y = 4x + 11$

iv) $2y = 3x + 18$ v) $2x + 3y = 9$ vi) $y = -\frac{1}{4}x - 7$

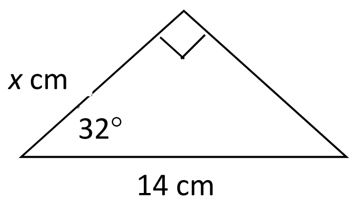
13. Find the equation of the straight line, in the form $ax + by + c = 0$

a) that passes through the point (3, 8) and is parallel to the line x – 2y + 7 =0

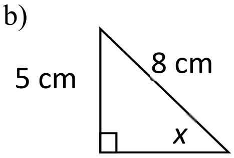
b) that is the perpendicular bisector of the line through (1, -5) and (-7, 11)

14. Find the value of x for each of the following. Give your answer to a suitable degree of accuracy.

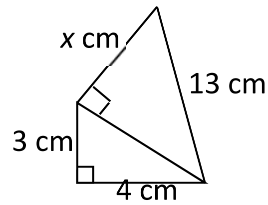
a)



b)

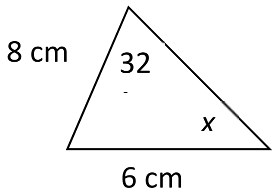


c)

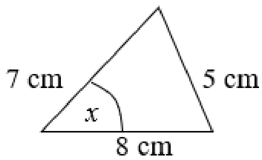


d) Calculate the acute angle between the *x*-axis and the line from the origin to the point (4, 5)

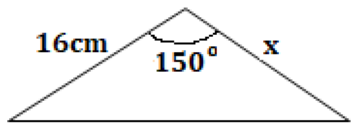
15. Calculate the angle marked x:



16. Calculate the value of the angle marked x.



17. The area of the following triangle is 56cm^2. Find the length marked x.



18. A square has vertices at (0, 0), (1, 0), (1, 1) and (0, 1). Graphs of the following equations are drawn on the same set of axes as the square.

$$x^2 + y^2 = 1, \qquad y = x + 1, \qquad y = -x^2 + 1, \qquad y = x, \qquad y = \frac{1}{x}$$

How many of the graphs pass through exactly two of the vertices of the square?

A 1 B 2 C 3 D 4 E 5

ANSWERS

1. a) x^4
   b) $32a^{10}b^{15}$
   c) $\frac{x^2z}{3} + \frac{3x}{4y^2}$

2. a) $n = 3$
   b) $x = 4$
   c) $x = \pm\frac{1}{8}$

3. a) $\sqrt{2}$
   b) $30\sqrt{10}$
   c) $4\sqrt{5}$
   d) $\frac{11\sqrt{7}-22}{3}$
   e) $-\frac{31+12\sqrt{3}}{23}$

4. a) $-4x^2 + 27x + 7$
   b) $14 + 6\sqrt{5}$
   c) $3a^2(4a^3 - 1)$
   d) $(2x - 5)(2x + 5)$
   e) $a = 3,\ b = -2,\ c = 7$

5. a) $x = 5,\ x = -7$
   b) $x = 0,\ x = 7$
   c) $x = -\frac{3}{2},\ x = \frac{1}{3}$

6. $x = \frac{3\pm\sqrt{19}}{2}$

7. a) $(-3, -20)$
   b) $2(x - 2)^2 + 15$
   c) $-\left(x - \frac{7}{2}\right)^2 + \frac{109}{4}$

8. a) $x = 4,\ y = 1$
   b) $x = 2,\ y = 3$
   c) $x = 5,\ y = 3\ ;\ \ x = 3,\ y = 5$

9. a) $x > 4.5$
   b) $9 \leq x \leq \frac{56}{3}$
   c) $x < -\frac{4}{3}\ ,\ x > \frac{1}{2}$

10. a) $\frac{7}{8}b$
    b) $\frac{7x-1}{(3x+2)(x-5)}$
    c) $x = 4,\ \ x = \frac{3}{7}$

11. a) 5
    b) $-\frac{3}{4}$
    c) $\left(-1,\ \frac{5}{2}\right)$

12. a) $parallel$: (i) and (iii)
    b) $perpendicular$: (iv) and (v)

13. a) $y = \frac{1}{2}x + \frac{13}{2}$
    b) $y = \frac{1}{2}x + \frac{9}{2}$

14. a) $x = 11.9\ cm$
    b) $x = 38.7°$
    c) $x = 12\ cm$
    d) 51.3°

15. $x =$ 45.0° $(not\ 45°, but\ allow\ 44.96°)$

16. $x = 38.2°$

17. $x = 14\ cm$

18. C

Let O = (0,0), A = (1,0), B = (1,1), C = (0,1) be the vertices of the square.
The equation $x^2 + y^2 = 1$ gives a circle passing through A and C.
The equation $y = x + 1$ gives a straight line passing only through C.
The equation $y = -x^2 + 1$ gives a parabola passing through A and C.
The equation $y = x$ gives a straight line passing through O and B.
The equation $y = \frac{1}{x}$ gives a rectangular hyperbola which has two branches and passes only through B.
So, only $x^2 + y^2 = 1$, $y = x$ and $y = -x^2 + 1$ have graphs passing through exactly two of the vertices of the square.